$\{X(t), t \geq 0\}$ Brownian motion $\sigma^2 = 1$

Let $t_1 < \dots < t_n$

Distribution of $\begin{pmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{pmatrix}$

General result, if $\underset{\sim}{U}, \underset{\sim}{V} \in \mathbb{R}^n$ g is 1-1 transformation

$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$

$f_{\underset{\sim}{U}}(\underset{\sim}{u}) = f_{\underset{\sim}{V}}(g^{-1}(\underset{\sim}{u})) \cdot |J|$ $|J|$ is Jacobian of the inverse transformation

Let $\underset{\sim}{U} = \begin{pmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & & 0 \\ 1 & 1 & 1 & 0 \\ 1 & \cdots & & 1 \end{pmatrix} \begin{pmatrix} X(t_1) \\ X(t_2) - X(t_1) \\ \vdots \\ X(t_n) - X(t_{n-1}) \end{pmatrix} = G \cdot \underset{\sim}{V}$

$G^{-1}\underset{\sim}{U} = G^{-1}\begin{pmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & 0 \cdots & 0 \\ 0 & \cdots & 0 \; -1 & 1 \end{pmatrix} \begin{pmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} X(t_1) \\ X(t_2) - X(t_1) \\ X(t_n) - X(t_{n-1}) \end{pmatrix}$

$f(x_1, \dots x_n) = f_{t_1}(x_1) \cdot f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$

$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \exp\left(-\frac{1}{2}\frac{x_1^2}{t_1}\right) \cdot \exp\left(-\frac{1}{2}\frac{(x_2 - x_1)^2}{t_2 - t_1}\right) \cdots \exp\left(-\frac{1}{2}\frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}\right)$

Concider $n = 2$. $(X(s), X(t))'$

$E[X(t)] = E[X(s)] = 0$

$Var(X(t)) = t$, $Var(X(s)) = s$

$cov(X(t), X(s)) = E[X(t) \cdot X(s)] = \underbrace{E[(X(t) - X(s))X(s)]}_{=0} + E[X(s)^2] = s$

Hence $\begin{pmatrix} X(s) \\ X(t) \end{pmatrix} \sim N_2\left(0, \begin{pmatrix} s & s \\ s & t \end{pmatrix}\right)$

$E[X(s) | X(t)] = \overset{E[X(s)]}{0} + \frac{cov(X(s), X(t))}{Var(X(t))}[X(t) - \overset{E[X(t)]}{0}] = \frac{s}{t} \cdot X(t)$

$Var(X(s) | X(t)) = Var(X(s)) - \frac{cov(X(s), X(t))^2}{Var(X(t))} =$

$s - \frac{s^2}{t} = \frac{s}{t}(t - s)$

Here we have used

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ & \Sigma_Y \end{pmatrix}\right)$

$E[X | Y] = \mu_X + \Sigma_{XY}\Sigma_Y^{-1}(Y - \mu_Y)$

$Var(X | Y) = \Sigma_X - \Sigma_{XY}\Sigma_Y^{-1}\Sigma_{YX}$

Application: bicycle race $Y(t)$: amount of time racer I, in the favorable position, is ahead 100% of completed, $Y(t)$ Brownian motion is a reasonable model.

Question 1: Racer I σ seconds ahead at $t = 1/2$ and wins.

?

$P(Y(1) > 0 | Y(\frac{1}{2}) = \sigma) =$

$P(Y(1) - Y(\frac{1}{2}) > -\sigma \mid Y(1/2) = \sigma) =$

$P(Y(1) - Y(\frac{1}{2}) > -\sigma) = P\left(\frac{Y(1/2)}{\sigma\sqrt{1/2}} > -\sqrt{2}\right) = \Phi(\sqrt{2})$

≈ 0.92

Question 2: Probability that racer I is ahead at $t = 1/2$ if racer I was the winner by σ seconds.

$P(Y(\frac{1}{2}) > 0 | Y(1) > \sigma) =$

Let $X(t) = Y(t)/\sigma$. Then $E[X(s) | X(t) = c/\sigma]$

$= \frac{cov(X(s), X(t))}{Var(X(t))} \cdot \frac{c}{\sigma} = \frac{s}{t} \cdot \frac{c}{\sigma}$

$Var(X(s) | X(t) = \frac{c}{\sigma}) = \frac{s(t-s)}{t}$

so $Y(s) = \sigma \cdot X(s) | Y(t) = c \sim N\left(\frac{sc}{t}, \frac{\sigma^2 s(t-s)}{t}\right)$

and $P(Y(\frac{1}{2}) > 0 | Y(1) = \sigma) = P(N(\frac{\sigma}{2}, \frac{\sigma^2}{4}) > 0)$

$= \Phi(1) \approx 0.84$

10.2 Hitting times, maximum variable and gambler's ruin problem.

$X(0) = 0$, $a > 0$

Define $T_a = \inf\{t : X(t) = a\}$

(1) $P(T_a < t)$?

$P(X(t) > a) = P(X(t) \geq a | T_a \leq t) \cdot P(T_a \leq t)$

$+ \underset{0}{P(X(t) \geq a | T_a > t)} \cdot P(T_a > t)$

Also $P(X(t) \geq a | T_a < t) = \frac{1}{2}$ since if hitting a before t it is equally likeli to be above or below a at time t.

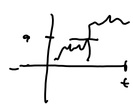
Thus $P(T_a \leq t) = 2 \cdot P(X(t) \geq a) =$

$\frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{1}{2}\frac{x^2}{t}} dx = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-\frac{x^2}{2}} dx$

If $a < 0$ $T_a \sim T_{-a}$ by symmetry

Hence $P(T \leq t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-y^2/2} dy$

(2) $P(\max_{0 \leq s \leq t} X(s) \geq a) = P(T_a \leq t)$ because $\{X(t), t \geq 0\}$ is continous.



$= 2 \cdot P(X(t) \geq a)$

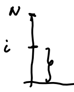
$= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2} dy$

when $a > 0$.

(3) If $A, B > 0$, $P(T_A < T_{-B})$

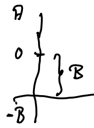
Use the symmetric random walk approximation to Brownian motion.

Gamblers ruin problem $p = 1/2$



P_i: starting with i units fortune and eventually reaching N, i.e. before hitting 0

$P_i = i/N$



In approximation step size Δx there are a "fortune" of size $B\Delta x$ and $(A+B)\Delta x$ is corresponding to N. Thus $P_i = \frac{i}{N} = \frac{B\Delta x}{(A+B)\Delta x} = P(T_A < T_{-B})$

10.3 Variations of Brownian motion.

We will consider two.

(1) Brownian motion with drift coefficient μ and variance σ^2 is defined by

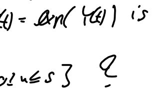
(i) $X(0)$

(ii) $\{X(t), t \geq 0\}$ has stationary increments

(iii) $X(t) \sim N(\mu t, t\sigma^2)$

Alterly: If $B(t)$ is standard Brownian motion

$X(t) = \mu t + \sigma B(t)$.



(2) Geometric Brownian motion.

If $\{Y(t), t \geq 0\}$ is a Brownian motion with drift μ and variance σ^2 then $X(t) = \exp(Y(t))$ is a geometric Brownian motion

If $s < t$, $E[X(t) | X(u), 0 \leq u \leq s]$?

$E[X(t) | X(u), 0 \leq u \leq s] = E[e^{Y(s) + Y(t) - Y(s)} | X(u), 0 \leq u \leq s]$

$= e^{Y(s)} \cdot E[e^{Y(t) - Y(s)} | Y(u), 0 \leq u \leq s]$

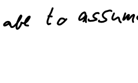
$= X(s) \cdot E[e^{Y(t) - Y(s)}]$

by the independent stationary increments

But $E[e^{Y(t) - Y(s)}] = \exp(\mu(t-s) + (t-s) \cdot \sigma^2/2)$

since W normal $\Rightarrow$ $E e^{aW} = \exp(a \cdot E(W) + a^2 \cdot Var(W)/2)$

So $E[X(t) | X(u), 0 \leq u \leq s] = X(s) \cdot \exp((t-s) \cdot (\mu + \sigma^2/2))$



Modeling stock prices

X_n price time n

$\left[1 + \frac{X_n - X_{n-1}}{X_{n-1}}\right] = \frac{X_n}{X_{n-1}} = Y_n$, Reasonable to assume

the Y_i, $i = 1, 2,$ are i.i.d

$X_n = Y_n \cdot X_{n-1} = Y_n \cdot Y_{n-1} X_{n-2} = Y_n Y_{n-1} \cdots Y_1 X_0$

so $\log(X_n) = \sum_{i=1}^n \log(Y_i) + \log(X_0)$

so approximating with a Brownian motion $\log(X_n)$ suitably normalized is approximated by Brownian motion with drift and X_n is approximated by a geometric Brownian motion with drift.