jan. 14-10:38

A stochastic process is a collection of random variables $\{X(t) : t \in T\}$

T is the index set, often time

$X(t)$ is the state of the process at time t

The following classification is possible

		State $X(t)$ discrete and countable	State $X(t)$ continuous
Index	countable	a	b
	real line	c	d

jan. 14-10:43

We will mainly consider a and c, shortly d (Brownian motion, but not d)

Markov chains

Here state is countable, time discrete so the process is of type $\{X_n, n \geq 0\}$ Typically state space is $\{0, 1, \dots\}$

4.1 Introduction

Simplest dependency between random variables is independent random variables. Then there is no information about X_{n+1} given that $X_n = x_n$

Let $\{X_n, n = 0, 1, 2, \dots\}$ be a stochastic process with state space $\{0, 1, 2, \dots\}$

Suppose the probability that if $X_n = i$ there is a fixed probability P_{ij} that $X_{n+1} = j$

Definition If $P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots X_0 = i_0) = P_{ij}$

for all $i, j, i_0, \dots, i_n$ and $n \geq 0$

the process is a Markov chain

jan. 14-10:48

The probabilities P_{ij} satisfy

$1 \geq P_{ij} \geq 0, \ \forall i, j \geq 0 \quad \sum_{j=0}^{\infty} P_{ij} = 1$

The transition matrix P

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & P_{00} & P_{01} & P_{02} \ \cdots \\ 1 & P_{10} & P_{11} & P_{12} \ \cdots \\ 2 & \vdots & \vdots & \end{array} \qquad P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Example Simple weather forecast

rain today probability α for rain tomorrow $1-\alpha$

no — " — — " — β — " — $1-\beta$

can be described by transition matrix

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Example Mood of Gary

cheerful C, so-so S, glum G

$$P = \begin{array}{c} \\ C \\ S \\ G \end{array} \begin{array}{c} \begin{array}{ccc} C & S & G \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

jan. 14-11:00

Example: Transforming into a Markov chain (higher order Markov chains)

$P($rain tomorrow $\mid$ rain today, rain yesterday$) = 0.7$

$P($ — " — $\mid$ — " —, not rain yesterday$) = 0.5$

$P($ — " — $\mid$ no rain today, rain yesterday$) = 0.4$

$P($ — " — $\mid$ no — " —, no rain yesterday$) = 0.2$

states: $\{$first day, second day$\} = \{RR, DR, RD, DD\}$

Possible transitions

$RR < \begin{matrix} RR \\ RD \end{matrix}$ $\quad P = 0.7$, $P = 0.3$

$DR < \begin{matrix} RR \\ RD \end{matrix}$ $\quad P = 0.5$, $P = 0.5$

$RD < \begin{matrix} DR \\ DD \end{matrix}$ $\quad P = 0.4$, $P = 0.6$

$DD < \begin{matrix} DR \\ DD \end{matrix}$ $\quad P = 0.2$, $P = 0.8$

$$P = \begin{array}{c} \\ RR \\ DR \\ RD \\ DD \end{array} \begin{array}{c} \begin{array}{cccc} RR & DR & RD & DD \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array}$$

jan. 14-11:26

Example Random walk model M

State space $Z = \{0, \pm 1, \pm 2, \dots\}$

$P_{i,i+1} = p = 1 - P_{i,i-1} = 1-p$

$$P = \begin{array}{c|ccccc} & \cdots -2 & -1 & 0 & 1 & 2 \cdots \\ \hline \vdots & & & & & \\ -2 & 0 & p & 0 & 0 & 0 \\ -1 & (1-p) & 0 & p & 0 & 0 \\ 0 & 0 & (1-p) & 0 & p & 0 \\ 1 & 0 & 0 & (1-p) & 0 & p \\ 2 & & & & & \\ \vdots & & & & & \end{array}$$

Example Gambling model

Each play $< \begin{matrix} \text{win } \$1 \text{ with probability } p \\ \text{lose } \$1 \quad \text{— " —} \quad 1-p \end{matrix}$

Game ends when gambler broke or fortune N

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 \cdots & & N-1 & N \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 \cdots & & 0 & 0 \\ 1-p & 0 & p \cdots & & \\ 0 & 1-p & 0 & p & \\ & & & & \\ 0 & \cdots & 1-p & 0 & p \\ 0 & \cdots & & & 1 \end{array} \right] \end{array}$$

state 0 and N are called absorbing

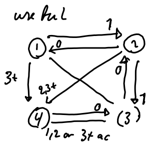
jan. 14-11:26

Example bonus / malus
State space 0, 1, 2, 3, transitions determined
by number of claims
number of claims is Poisson distributed

$$P(\#\text{ claims}) = \frac{\lambda^k}{k!} e^{-\lambda} = a_k, \quad k = 0, 1, 2$$

$a_3 = 1 - a_0 - a_1 - a_2$
i.e. more than two claims

Graph useful



$$P = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 = 1 - a_0 - a_1 - a_2 \\ a_0 & 0 & a_1 & a_2 + a_3 = 1 - a_1 - a_0 \\ 0 & a_0 & 0 & 1 - a_0 \\ 0 & 0 & a_3 & 1 - a_0 \end{bmatrix}$$

Summary: The distribution of a Markov chain $\{X_n, n \geq 0\}$ is determined by the initial distribution $P(X_0 = i)$ $i = 0, \dots$
and the transition probabilities

$P(X_{n+1} = j \mid X_n = i)$ $n = 1, 2, \dots$

and the transition probabilities are independent of $X_0, \dots, X_{n-1}$
What happens in the future $X_n, X_{n+1}, \dots$
is independent of the past $X_0, X_1, \dots, X_{n-1} = x_{n-1}$
given the present $X_n = x_n = i$.

jan. 14-11:26