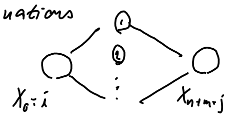
- $\{X_n, n \geq 0\}$ Markov chain Section 4.2

P_{ij} one-step transitions probabilities

<u>Definition</u> The n-step transition probabilities $n \geq 0$ are $P_{ij}^{(n)} = P(X_{n+k} = j \mid X_k = i)$ $n \geq 0$ $\forall$ i, j

The Chapman-Kolmogorov equations

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$



Formally:

$X_n = k$

$$P_{ij}^{(n+m)} = P(X_{n+m} = j \mid X_0 = i)$$
$$= \sum_{k=0}^{\infty} P(X_{n+m} = j, X_n = k, X_0 = i) / P(X_0 = i)$$
$$= \sum_{k=0}^{\infty} P(X_{n+m} = j \mid X_n = k, X_0 = i) \cdot P(X_n = k \mid X_0 = i)$$
$$= \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$

Result: $P^{(n)} = P^n$ i.e. n'th step transition matrix is the same as the matrix product of n one-step ahead transition matrices

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Proof. (by induction)

$n = 2$ $P^{(2)} = P^{(1+1)} = P^{(1)}P^{(1)} = PP = P^2$

If $P^{(n-1)} = P^{n-1}$ then $P^{(n)} = P^{(n+1-1)}$

$= P^{(n-1)}P^{(1)} = P^{n-1} \cdot P^1 = P^n$

Remark 4.2.1 Now we have a method for finding the n-step transition probabilities.

Example wether forecast

$$P = \begin{array}{c} R \\ D \end{array} \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

Probability that it will rain on the day four days ahead from today if it rains today

P_{11}^4 ? $P^2 = \begin{pmatrix} 0.61 & 0.39 \\ & \end{pmatrix}$

$P^4 = (P^2)^2$

Example weather forecast 4 states

RR DR RD DD

P(rain Thursday | rain Monday and Tuesday) ?

P(rain Thursday | rain Monday and Tuesday)

= P(rain Thursday and Wednesday | rain Monday and Tuesday)

+ P(rain Thursday and dry Wednesday | — " —)

But rain Thursday, rain Wednesday → state 0 RR

— " — dry — " — → state 1 DR

Hence P(rain Thursday | rain Monday Tuesday)

$= P_{10}^2 + P_{00}^2$ where

$$P^2 = \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.5 & 0 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.49 & \cdot & 0.12 \cdots \\ \vdots & & \vdots \end{pmatrix}$$

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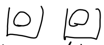
P(rain Thursday | rain Monday and Tuesday) = 0.49 + 0.12 = 0.61

Example Urn model



Two types of ball red and blue. The urns has one ball each

Remove one ball, add one with same color with p = 0.8. X_n = # red balls, start with two

X_n Markov chain

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

P(fifth selection red) ?

$$P(\text{fifth selection red}) = \sum_{i=0}^{2} P(\text{fifth selection red} \mid X_4 = i) \underbrace{P(X_4 = i \mid X_0 = 2)}_{P_{2i}^4}$$

$$= 0 \cdot P_{20}^4 + 0.5 P_{21}^4 + 1.0 P_{22}^4$$

Wanted probability 0.7048

Example 8 urns, balls randomly distributed

? → P(3 nonempty urns | 9 balls distributed)

X_n = # nonempty urns after n distributions

X_n Markov

$P_{ii} = i/8 = 1 - P_{i,i+1}$ $(i = 0, 1, \ldots, 8)$

P(3 nonempty urns | 9 balls distributed) $= P_{03}^9 = P_{13}^8$ since $P_{01} = 1$

Can collapse states 4, 5, 6, 7, 8 into one state

New transition probability matrix is

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Such "lumping" does not generally result in a new Markov Chain

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X_n Markov chain, transition matrix P, $X_0 = i \notin \mathcal{A}$

$\mathcal{A}$ set of states

$\alpha_1 = P(X_k \in \mathcal{A}$ some $k = 0, \ldots, m \mid X_0 = i)$

Idea: Define a new Markov chain with one ~~absorbing~~ absorbing set of states

$$N = \begin{cases} \min\{n : X_n \in \mathcal{A}\} & \text{if } X_n \text{ ever is in } \mathcal{A} \\ \infty & X_n \notin \mathcal{A} \text{ all } n \end{cases}$$

$$W_n = \begin{cases} X_n & n < N \\ A & n \geq N \end{cases}$$ A absorbing state

W_n Markov chain with transition matrix

$Q_{ij} = P_{ij}$ $i, j \notin \mathcal{A}$

$Q_{iA} = \sum_{j \in \mathcal{A}} P_{ij}$ $i \in \mathcal{A}$

$Q_{AA} = 1$

Then $\alpha_1 = P(W_m = A \mid X_0 = i)$

$= P(W_m = A \mid W_0 = i)$

$= Q_{i,A}^m$

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Example Sequence of flips of fair coin

N = # trials until $\underbrace{\text{three consegutive Success}}_{\text{run of length 3}}$

P(N ≤ 8) ?

Define four states {0, 1, 2, 3}

$X_n = 0 \iff$ current run is F

$X_n = 1 \iff$ current run a success

$X_n = 2 \iff$ current run SS

$X_3 = 3 \iff$ SSS

Transition matrix

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1/2 & 0 \\ \frac{1}{2} & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Then $p = P_{03}^8$

$P(N = 8) = P(N \leq 8) - P(N \leq 7)$

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$\alpha_2 = P(X_m = j, X_k \in \mathcal{A}, k = 1, \ldots, m-1 \mid X_0 = i)$

$\{X_m = j, X_k \notin \mathcal{A}\ k = 1, \ldots, m-1\} \iff W_m = j$

So $\alpha_2 = P(W_m = j \mid X_0 = i)$

$= P(W_m = j \mid W_0 = i) = Q_{ij}^m$

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