n-step transition probabilities

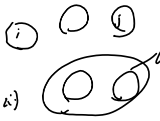
$P^{(n)}_{ij}$, From now on P^{n}_{ij}

Similarly for matrices P^{n} instead of $P^{(n)}$

$\mathcal{A}$

W_n Markov chain transition matrix $Q = \{Q_{ij}\}$. W_n is identical to chain X_n up to $X_n = j$, $j \in \mathcal{A}$. From then on $W_n = A$, A absorbing state

1. Considered situation when probability to entering $\mathcal{A}$ some time before k called probability α_1



$\mathcal{A}$

$$\alpha_2 = P(X_m = j, X_k \notin \mathcal{A}\ k = 1, \dots, m-1 \mid X_0 = i)$$

$$\{X_m = j, X_k \notin \mathcal{A}\ k = 1, \dots, m-1\} \iff W_m = j$$

so $\alpha_2 = P(W_m = j \mid X_0 = i) = P(W_m = j \mid W_0 = i) = Q^{m}_{ij}$

jan. 21-10:14

Example $p = P(X_4 = 2, X_3 \le 2, X_2 \le 2, X_1 \le 2 \mid X_0 = 1)$

States $\{1,2,3,4,5\}$, p?

$p = P(X_4 = 2, X_i \notin \{3,4,5\}\ i = 1,2,3 \mid X_0 = 1)$

$$Q = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & P_{11} & P_{12} & 1 - P_{11} - P_{12} \\ 2 & P_{21} & P_{22} & 1 - P_{21} - P_{22} \\ 3 & 0 & 0 & 1 \end{array}$$

$p = Q^{4}_{12} = P(W_4 = 2 \mid W_0 = 1)$

second state of W_n

3.

$i \notin \mathcal{A}$

$j \in \mathcal{A}$

$\mathcal{A}$

Let $\alpha_3 = P(X_m = j, X_k \notin \mathcal{A}, k = 1, \dots, m-1 \mid X_0 = i)$

$$\alpha_3 = \sum_{r \notin \mathcal{A}} P(X_m = j, X_{m-1} = r, X_k \notin \mathcal{A}\ k = 1, \dots, m-2, X_0 = i) / P(X_0 = i)$$

$=$

jan. 21-10:28

$$\sum_{r \notin \mathcal{A}} P(X_m = j \mid X_{m-1} = r, X_k \notin \mathcal{A}, k = 1, \dots, m-1, X_0 = i) \times$$

$$P(X_{m-1} = r, X_k \notin \mathcal{A}, k = 1, \dots, m-1 \mid X_0 = i) \cdot P(X_0 = i) / P(X_0 = i)$$

$$= \sum_{r \notin \mathcal{A}} P_{rj}\, Q^{m-1}_{ir}$$

so what is going on is that chain starts in $i \notin \mathcal{A}$, stays out of $\mathcal{A}$ and then jumps to $j \in \mathcal{A}$

$i \in \mathcal{A}$

$$\alpha_4 = P(X_m = j, X_k \notin \mathcal{A}, k = 1, \dots, m-1 \mid X_0 = i)$$

$$= \sum_{r \notin \mathcal{A}} P(X_m = j\ X_k \notin \mathcal{A}\ k = 1, \dots, m-1 \mid X_0 = i\ X_1 = r) \cdot P(X_1 = r \mid X_0 = i)$$

$$= \sum_{r \notin \mathcal{A}} P(X_{m-1} = j, X_k \notin \mathcal{A}, k = 1, \dots, m-2 \mid X_1 = r) \cdot P_{ir}$$

where we have used the Markov property

For $i \in \mathcal{A}$, $j \notin \mathcal{A}$ this reduces to

$$\sum_{r \notin \mathcal{A}} Q^{m-1}_{rj}\, P_{ir}$$

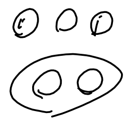
jan. 21-10:39

so here chain starts in i, jumps out of $\mathcal{A}$ and stays there until $X_m = j$

$i, j \notin \mathcal{A}$

$$\alpha_5 = P(X_n = j \mid X_0 = i, X_k \notin \mathcal{A}, k = 1, \dots, n)$$

$$= \frac{P(X_n = j, X_k \notin \mathcal{A}, k = 1, \dots, n \mid X_0 = i)}{P(X_k \notin \mathcal{A}, k = 1, \dots, n \mid X_0 = i)}$$



i.e. Here one conditions on $X_k \notin \mathcal{A}$ $k = 1, \dots, n$ and on $X_0 = i$ while in situation with α_2 one only conditioned on $X_0 = i$

$$\alpha_5 = \frac{Q^{n}_{ij}}{\sum_{r \notin \mathcal{A}} Q^{n}_{ir}}$$

jan. 21-10:52

Remark 4.2.2 Up to now we have considered conditional probabilities $P(X_1 = j \mid X_0 = i) = P_{ij}$

The unconditional probabilities for an event is computed as follows

$$P(X_n = j) = \sum_{i=0}^{\infty} P(X_n = j, X_0 = i)$$

$$= \sum_{i=0}^{\infty} P(X_n = j \mid X_0 = i) \cdot P(X_0 = i)$$

so to get $P(X_n = j)$ one has to specify $P(X_0 = i)$

i.e to get the unconditional probability the probability for the initial state must be specified

jan. 21-11:19

4.3 Classification of states.

Definition: State j is <u>accessible</u> from state i is $P^{n}_{ij} >$ for some n.

If j is not accessible from state i it not possible to enter j starting from i.

Definition Two states which are accessible to each other are said to <u>communicate</u>, written $i \leftrightarrow j$.

Properties of communicating states

(i) state i communicates with itself since $P(X_0 = i \mid X_0 = i) = 1$

(ii) state i communicates with state j then state communicates with state i.

(i) & (ii) follows from the definition

(iii) if state i communicates with state j and state j communicates with state k then state i communicates with state k

Formally $P^{n+m}_{ik} = \sum_{r=0}^{\infty} P^{n}_{ir} P^{m}_{rk} \ge P_{ij} P_{jk}$

jan. 21-11:26

Definition. Two states that communicate is in the <u>same class</u>

It follows (i),(ii) and (iii) that two classes are either identical or disjunct

Definition. A Markov chain is irreducible if there is only one class

Example

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix} \end{array}$$

P is irreducible since all states communicate

e.g $0 \to 1 \to 2$

Example

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/2 & 1/2 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Here classes are $\{0,1\}$ $\{2\}$ and $\{3\}$

Let $f_i = P($ there is some $i > 0$ where $X_n = i \mid X_0 = i)$

$= P(X_n$ returns to $i \mid X_0 = i)$

Definition: A state is recurrent if $f_i = 1$

A state is transient if $f_i < 1$

jan. 21-11:36

Due to the Markov property if state i is recurrent and the process enters i it will start afresh with initial state i.

So it will enter i infinitely often.

If the state is transient there will be a probability $1 - f_i$ that it will never enter state i again. Hence

$$P(X_n = i, k \text{ times}) = f_i^{k-1}(1 - f_i) \qquad k = 1, 2, 3, \dots$$

Define $I_n = \begin{cases} 1 & X_n = i \\ 0 & \text{else} \end{cases}$

Then $\sum_{n=0}^{\infty} I_n$ is the number of times $X_n = i$

Hence $E\left[\sum_{n=1}^{\infty} I_n\right] = \sum_{n=1}^{\infty} E\, I_n$

$$= \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) = \sum_{n=1}^{\infty} P_{ii}^n$$

Proposition State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

State i is transicient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

jan. 21-11:50