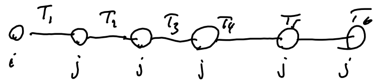
Markov chain X_n irreducible and recurrent



Total length $T_1 + T_2 + \dots + T_6$ $\quad 6 = n$

visits to j is 6 $\quad 6 = n$

Proportion of transitions in j $\quad \frac{n}{T_1 + \dots + T_n} = \frac{1}{\frac{T_1}{n} + \frac{T_2 + \dots + T_n}{n}}$

assume that $E(T_i) = m_j$, $i = 2, \dots$ $\quad \rightarrow \begin{cases} 1/m_j & \text{when } m_j < \infty \\ \infty & \text{if } m_j = \infty \end{cases}$

So $\pi_j = 1/m_j$. Also note that the initial state i does not matter

Remark 4.4.1 j is positive recurrent $\Leftrightarrow m_j < \infty$

j is null recurrent if $m_j = \infty$

Since $\pi_j = 1/m_j$, positive recurrence means $\pi_j > 0$, null recurrence $\pi_j = 0$

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Will show that positive recurrence is a class property

Proposition 4.5 Let i be positive recurrent and $i \leftrightarrow j$ then j is positive recurrent

Proof Let i be positive recurrent

Since $i \leftrightarrow j$, $P_{ij}^n > 0$ for some n

$\pi_i P_{ij}^n$ is the long proportion of time/transitions that chain is in j and was in i n transitions earlier.

π_j long run proportion the chain is in j

Thus $\quad 0 < \pi_i P_{ij}^n \le \pi_j$

Hence $\pi_j > 0$ and $m_j = 1/\pi_j < \infty$ ∎

Remark 4.4.1 Also null recurrence is a class property. Assume i null recurrent and $i \leftrightarrow j$

i recurrent and $i \leftrightarrow j \Rightarrow$ j is recurrent

If j were posi. recurrent, then Prop 4.5 $\Rightarrow$ i positive recurrent which is a contradiction.

Hence j null recurrent

Remark 4.4.2 An irreducible finite state Markov chain is positive recurrent. If it were null recurrent all states would have long-run proportions of transitions equal to 0.

Remark 4.4.3 $\pi_i P_{ij}$ is the long run proportion of transitions from i to j.

Hence $\pi_j = \sum_{i \in \{\text{possible states}\}} \pi_i P_{ij}$

General theorem:

Theorem 4.1 Let X_n be an irreducible Markov Chain

Then (a) If X_n is positive recurrent the long-run proportions are <u>unique</u> and solves

$\pi_j = \sum_i \pi_i P_{ij} \quad \forall j \quad \sum \pi_j = 1, \qquad \forall =$ "for all"

(b) If there is no solution of the equation then $\pi_j = 0 \ \forall j$ and chain is either transient or null recurrent

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Example Weather forecast

$$P = \begin{array}{c c} & \begin{array}{cc} R & D \end{array} \\ \begin{array}{c} R \\ D \end{array} & \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

$(\pi_R, \pi_D) = (\pi_R, \pi_D)\begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$

$\pi_R = \pi_R \cdot \alpha + \pi_D \beta$

$\pi_D = (1-\alpha)\pi_R + (1-\beta)\pi_D$

$\pi_R + \pi_D = 1$

$\pi_R = \alpha \cdot \pi_R + \beta(1 - \pi_R) \Rightarrow \pi_R = \frac{\beta}{1+\beta-\alpha}, \ \pi_D = \frac{1-\alpha}{1+\beta-\alpha}$

Example

$$P = \begin{array}{c c} & \begin{array}{ccc} C & S & G \end{array} \\ \begin{array}{c} C \\ S \\ G \end{array} & \begin{array}{ccc} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{array} \end{array}$$

$\boxed{(\pi_c, \pi_s, \pi_G) = (\pi_c, \pi_s, \pi_G) P}$

which gives equations

$\pi_c = 0.5\pi_c + 0.3\pi_s + 0.2\pi_G$

$\pi_s = 0.4\pi_c + 0.4\pi_s + 0.3\pi_G$

$\pi_G = 0.1\pi_c + 0.3\pi_s + 0.5\pi_G$ also $\pi_c + \pi_s + \pi_G = 1$

$\pi_c = \frac{21}{62}, \ \pi_s = \frac{23}{62}, \ \pi_G = \frac{18}{62}.$

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