$\pi = (\pi_0, \dots, \pi_k)$ long run proportions

$\pi = \pi P$, $\sum \pi_j = 1$

Example Hardy-Weinberg law and Markov chains

Consider a large population

Pool of genes of types a and A

Each individual has two genes.

Consider a first generation

Types of combinations	proportions	
AA	p_0	$p_0 + q_0 + r_0 = 1$
aa	q_0	
Aa	r_0	

Random mating: Individuals are selected randomly and contribute one gene each to new individual, and this gene picked randomly

In the next generation the proportion of genes are

$$P(A) = P(A|AA)\cdot P(AA) + P(A|aa)\cdot P(aa) + P(A|Aa)\cdot P(Aa) = 1\cdot p_0 + 0\cdot q_0 + \tfrac{1}{2}\cdot r_0 = p_0 + r_0/2$$

$$P(a) = q_0 + r_0/2$$

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The proportion of gene combinations are now

$$p = P(AA) = P(A)P(A) = \left(p_0 + \tfrac{r_0}{2}\right)^2$$

$$q = P(aa) = P(a)P(a) = \left(q_0 + \tfrac{r_0}{2}\right)$$

$$r = P(Aa) + P(aA) = 2P(A)\cdot P(a) = 2\cdot\left(p_0 + \tfrac{r_0}{2}\right)\left(q_0 + \tfrac{r_0}{2}\right)$$

because mating is random

In a third generation the proportion of A and a is $p + \frac{r}{2}$ and $q + r/2$

But $p + \frac{r}{2} = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$

$= \left(p_0 + \frac{r_0}{2}\right)\left[p_0 + \frac{r_0}{2} + q_0 + \frac{r_0}{2}\right] = \left(p_0 + \frac{r_0}{2}\right)$

So the proportion of genes of type A and a is the same as in the previous generation

Hence with random mating the proportion of individuals with combinations AA, aa, Aa will be fixed as p, q and r.

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This is the famous Hardy-Weinberg law.

π What about Markov Chains?

Consider a situation whereas individuals have 1 offspring and let X_n be the state at n'th generation. X_n is a Markov chain with transition matrix

$\boxed{\pi = \pi P}$

$$P = \begin{array}{c|ccc} & AA & aa & Aa \\ \hline AA & p + \frac{r}{2} & 0 & q + r/2 \\ aa & 0 & q + \frac{r}{2} & p + r/2 \\ Aa & \frac{1}{2}\left(p + \frac{r}{2}\right) & \frac{1}{2}\left(q + \frac{r}{2}\right) & \frac{p}{2} + \frac{q}{2} + \frac{r}{2} \end{array}$$

Is $\pi = (p, q, r)$ the long-run proportions?

$$(p, q, r)\begin{bmatrix} P_{00} \\ P_{10} \\ P_{20} \end{bmatrix} = p\left(p + \frac{r}{2}\right) + r\left(\frac{p}{2} + \frac{r}{4}\right) = \left(p + \frac{r}{2}\right)^2$$

which we have shown is equal to $P(AA)$ $= p$.

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similarly $(p, q, r)\begin{pmatrix} P_{01} \\ P_{11} \\ P_{12} \end{pmatrix} = q\left(q + \frac{r}{2}\right) + \frac{r}{2}\left(q + \frac{r}{2}\right) = \left(q + \frac{r}{2}\right)^2 = q = P(aa)$

and $(p, q, r)\begin{pmatrix} P_{02} \\ P_{12} \\ P_{22} \end{pmatrix} = p\left(q + \frac{r}{2}\right) + q\left(p + \frac{r}{2}\right) + \frac{r}{2}(p + q + r)$

$= 2pq + \frac{pr}{2} + \frac{qr}{2} + \frac{pr}{2} + \frac{qr}{2} + \frac{r^2}{2}$

$= 2pq + pr + qr + \frac{r^2}{2} = 2\left(p + \frac{r}{2}\right)\left(q + \frac{r}{2}\right) =$

$P(Aa) = r.$

Hence we have shown $\pi = \pi P$

where $\pi = (p, q, r)$ is Hardy-Weinberg law.

The long run proportions π_j, $j \geq 0$ is often called the stationary probabilities and satisfy the following. If

$\pi_j = P(X_0 = j)$ then $\pi_j = P(X_n = j)$ $n \geq 0$ and $j \geq 0$

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To show the result one can use induction $\boxed{\pi = \pi P}$

OK for $n = 0$, so suppose OK for $n-1$, i.e.

$\pi_j = P(X_{n-1} = j)$ $\forall j$. Then

$$P(X_n = j) = \sum_i P(X_n = j, X_{n-1} = i) = \sum P(X_n = j | X_{n-1} = i)\cdot \underbrace{P(X_{n-1} = i)}_{\pi_i}$$

$$= \sum_i \pi_i \cdot P_{ij} = \pi_j$$

Proposition 4.6 Let $\{X_n, n \geq 0\}$ be an irreducible Markov chain with stationary probabilities π_j, $j \geq 0$ and let r be a bounded function on the state space, then

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j=0}^{\infty} r(j)\cdot\pi_j$$

Proof Let $a_j(N)$ be the amount of time X_n is in j until period N

Then $\sum_{n=1}^{N} r(X_n) = \sum_{j=0}^{\infty} a_j(N)\cdot r(j)$. But $\frac{a_j(N)}{N} \Rightarrow \pi_j$

Hence $\lim \frac{1}{N}\sum_{n=1}^{N} r(X_n) = \lim \sum_{j=0}^{\infty} \frac{a_j(N)}{N} r(j) \to \sum_{j=0}^{\infty} r(j)\cdot\pi_j$ ∎

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Example Bonus/malus, auto insurance

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & a_0 & a_1 & a_2 & a_3 \\ 2 & a_0 & 0 & a_1 & 1 - a_0 - a_1 \\ 3 & 0 & a_0 & 0 & 1 - a_0 \\ 4 & 0 & 0 & a_0 & 1 - a_0 \end{array}$$

states $\{1, 2, 3, 4\}$

premiums 200, 250, 400, 600

$a_k = \frac{\lambda^k}{k!} e^{-\lambda}$

$k = 1, 2, 3$

stationary probability solves

$\pi_1 = a_0\pi_1 + a_0\pi_2$

$\pi_2 = a_1\pi_1 + a_0\pi_3$

$\pi_3 = a_2\pi_1 + a_1\pi_2 + a_0\pi_4$ $\pi_1 + \dots + \pi_4 = 1$

Average annual premium

$AAP = 200\cdot\pi_1 + 250\cdot\pi_2 + 400\cdot\pi_3 + 600\,\pi_4$

When $\lambda = 1/2$, $a_0 = 0.6065$, $a_1 = 0.3033$, $a_2 = 0.0758$

$\pi_1 = 0.3692$, $\pi_2 = 0.2395$, $\pi_3 = 0.2103$, $\pi_4 = 0.1809$

Then AAR 326.375

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4.4.1 Limiting probabilities

If $P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$, $P^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5638 & 0.4332 \end{pmatrix}$

$\pi P = \pi$

$P^8 = \begin{pmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{pmatrix}$, $\pi_1 = \frac{4}{7} = 0.571$

$\pi_2 = \frac{3}{7} = 0.429$

so will $P^n \to$ matrix with equal rows and with ~~stationary probabili~~ rows given by stationary probabilities

That is not always true.

If $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, X_n alternates between 0 and 1 and the long run proportions are $\pi_0 = 1/2$, $\pi_1 = 1/2$

Here $P_{00}^n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Definition: A chain that can only return to a state in multiple of $d > 1$ is called <u>periodic</u> and does not have limiting probabilities. If the chain is not periodic it is aperiodic.

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For aperiodic chains the limiting probabilities when they exist are equal to the long run proportions and are independent of initial states.

Let $\alpha_j = \lim_{n\to\infty} P(X_n = j)$

But $P(X_{n+1} = j) = \sum_{i=0}^{\infty} P(X_{n+1} = j \mid X_n = i) \cdot P(X_n = i)$

$= \sum_{i=0}^{\infty} P_{ij} \cdot P(X_n = i)$

and also $1 = \sum_{i=0}^{\infty} P(X_{n+1} = i)$

Let $n \to \infty$. Then $\alpha_j = \sum_{i=0}^{\infty} \alpha_i P_{ij}$

so $\{\alpha_i\}$ solves the equations for $\{\pi_j\}$, where they are unique solution for $\pi = \pi P$

Hence $\alpha_j = \pi_j$. A sufficient condition for uniqueness is that the chain is irreducible and positive recurrent

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