Consider only cases where CD_j changes
The test can be modelled as a Gambler's ruin problem
with $2M+1$ states with

$$p = \frac{P_1(1-P_2)}{P_1(1-P_2)+P_2(1-P_1)}$$ is the probability of increase

$q = 1-p$

Hence, M is reached first with probability

$$\frac{1-\left(\frac{q}{p}\right)^{M}}{1-\left(\frac{q}{p}\right)^{2M}} = \frac{1}{1+\left(\frac{q}{p}\right)^{M}} = P(D_1 \text{ best in test})$$

$N = 2M+1$ states

$$\underbrace{0, \dots, M-1}_{M},\ \underbrace{M}_{i},\ \underbrace{M+1, \dots, 2M+1}_{M}$$

feb. 6-10:26

4.6 Mean time spent in transient states

Rember: recurrent state: revisited infinitely often
transient state: revisited finite number of times.

Assume finite number of states, X_n Markov chain
Let $T = \{1, \dots, t\}$ be the set of transient states

$$P_T = \begin{pmatrix} P_{11} & \cdots & P_{1t} \\ P_{t1} & & P_{tt} \end{pmatrix}$$

Not possible that $\sum_{j=1}^{T} P_{ij} = 1$ for all i.

Consider situation when i and j transient
Let s_{ij} be the expected number of periods that X_n is in j given $X_0 = i$
Conditiong on the initial transitions which can be to a recurrent state

$$(*)\quad s_{ij} = \delta_{ij} + \sum_{k} P_{ik} s_{kj} \quad \text{where } \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

$$= \delta_{ij} + \sum_{k=1}^{t} P_{ik} s_{kj}$$

since $s_{kj} = 0$ if k is recurrent because X_n cannot move from a recurrent to a transient state so the expected number of such transitions is 0.

On matrix form the identeties $(*)$ can be written $S = I + P_T S$ where $S = \{s_{ij}\}_{i,j=1}^{t} = \begin{pmatrix} s_{11} & s_{1t} \\ s_{t1} & s_{tt} \end{pmatrix}$

feb. 6-10:38

$S = I + P_T S$ can be written as $(I - P_T)S = I$

Note that since I is non-singular, then $I - P_T$ and S must also be non-singular. Therefore $(I-P_T)^{-1}$ exists

and $S = (I - P_T)^{-1}$

Let $f_{ij} = P(X_n = j \text{ some } n > 0 \mid X_0 = i)$.
Also an expression for f_{ij} can be found by conditing on the being in j sometime.

$$s_{ij} = E[\text{time in } j \mid X_n = j \text{ some } n > 0, X_0 = i]\, f_{ij} \quad (**)$$
$$+ E[\text{time in } j \mid X_n \neq j \text{ all } n > 0, X_0 = i](1 - f_{ij})$$
$$= (\delta_{ij} + s_{jj}) f_{ij} + \delta_{ij}(1 - f_{ij})$$

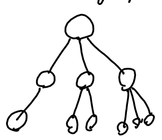
where s_{jj} is the expected number of additional periods spent in j given that X_n ever enter j.

Remark $(\delta_{ij} + s_{jj}) f_{ij} = \begin{cases} (1 + s_{ii}) f_{ii} & i = j \\ s_{jj} f_{ij} & i \neq j \end{cases}$

Solving gives $f_{ij} = \dfrac{s_{ij} - \delta_{ij}}{s_{jj}}$ $\quad i, j = 1, \dots, t.$

feb. 6-10:41

4.7 Branching processes



Given a population of individuals producing offspring. Each individual produce a number of offspring independently of the number others produce.
Let Z be the random variable describing number of offspring

$P(Z = j) = P_j$ where $P_j < 1$ all j.

and X_n be the number of individuals in the n'th generation. Then

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$ where X_0 is the initial number of individuals

X_n is a Markov chain and $P_{00} = 1$ so 0 is recurrent
also $P_{i0} = P_0^i$ so if $P_0 > 0$ there is a positive probability that a generation with i individuals will die out
A set of transient states $\{1, \dots, t\}$ will be visited only a finite number of times. Hence, if $P_0 > 0$ either the chain will die out or move to ∞.

feb. 6-11:35

Assume $X_0 = 1$ and let $\mu = E(Z)$, $\sigma^2 = Var(Z)$.

$$E[X_n] = E\left[\sum_{i=1}^{X_{n-1}} Z_i\right] = E\left[E\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}\right]$$
$$= E[X_{n-1} \cdot E(Z_i)] = \mu \cdot E[X_{n-1}].$$

Continuing the recursion

$$E(X_n) = \mu^n \cdot E(X_0) = \mu^n.$$

Similary for $Var(X_n)$.
Use $Var(X_n) = E[Var(X_n \mid X_{n-1})] + Var(E[X_n \mid X_{n-1}])$
where $Var(X_n \mid X_{n-1}) = Var\left(\sum_{i=1}^{X_{n-1}} Z_i \mid X_{n-1}\right) = \sigma^2 \cdot X_{n-1}$
$E[X_n \mid X_{n-1}] = \mu \cdot X_{n-1}$

so we get the recursion

$$\underline{Var(X_n)} = E[\sigma^2 X_{n-1}] + Var(\mu \cdot X_{n-1})$$
$$= \sigma^2 \mu^{n-1} + \mu^2 \cdot \underline{Var(X_{n-1})}$$

feb. 6-11:48

Solving the recursion

$$Var(X_n) = \sigma^2[\mu^{n-1} + \mu^n + \dots + \mu^{2n-2}] + \mu^{2n}\underbrace{Var(X_0)}_{0}$$

Hence $Var(X_n) = \begin{cases} \sigma^2 \mu^{n-1} \cdot \dfrac{1-\mu^n}{1-\mu} & \mu \neq 1 \\ n \cdot \sigma^2 & \mu = 1 \end{cases}$

So $Var(X_n) \to \begin{cases} \infty & \mu > 1 \\ 0 & \end{cases}$

feb. 6-11:56