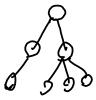
1 Branching processes.
Models for dynamics of population

$X_0 = 1$



X_n size of population time n

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \quad Z_i \text{ offspring distribution}$$

$\mu = \bar{Z}$, $\sigma^2 = Var(Z)$

$P(Z=j) = P_j$

Have shown: If $P_0 > 0$ either $X_n \to 0$ or $X_n \to \infty$

Let $\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1)$ i.e. the probability that the population dies out

π_0 ?

$\underline{① \ \mu < 1}$: $\mu^n = E(X_n) = \sum_{j=1}^{n} j\, P(X_n = j \mid X_0 = 1)$

$\geq \sum_{j=1}^{n} P(X_n = j \mid X_0 = 1)$

$P(X_n \geq 1 \mid X_0 = 1)$

Since $\mu^n \to 0$, then

$$\lim P(X_n = 0 \mid X_0 = 1) = 1 - \underbrace{\lim P(X_n \geq 1 \mid X_0 = 1)}_{\to 0}$$

and $\lim P(X_n = 0 \mid X_0 = 1) \to 1$.

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$\mu > 1$. $\pi_0 = P(\text{population dies out} \mid X_0 = 1)$

$$= \sum_{j=0}^{\infty} P(\text{population dies out} \mid X_1 = j) \cdot \underbrace{P(X_1 = j \mid X_0 = 1)}_{P(Z=j) = P_j}$$

$$= \sum_{j=0}^{\infty} \pi_0^{\,j} P_j$$

Then π_0 is the smallest value that satisfies $\pi_0 = \sum_{j=0}^{\infty} \pi_0^{\,j} P_j$

$\mu = 1$: Then one can show $\lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1) \to 1$

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Examples: $P_0 = 1/2$, $P_1 = 1/4$, $P_2 = \frac{1}{4}$

Then $\mu = 0 \cdot 1/2 + 1 \cdot 1/4 + 2/4 = 3/4 < 1$.

Example $P_0 = 1/4$, $P_1 = 1/4$, $P_2 = 1/2$

$\mu = 1 \cdot 1/4 + 2 \cdot 1/2 = 5/4 > 1$.

π_0 satisfies

$\pi_0 = \frac{1}{4} + \frac{1}{4}\pi_0 + \frac{1}{2}\pi_0^2$

so $2\pi_0^2 - 3\pi_0 + 1 = 0$ with solution

$$\frac{3 \pm \sqrt{9-8}}{2 \cdot 2} = \begin{cases} 1 \\ 1/2 \end{cases}$$

so $\pi_0 = 1/2$.

Example If $X_0 = n$ then $\lim P(\text{population dies out}) = \pi_0^{\,n}$.

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4.8 Time reversible Markov chains

Suppose that X_n is ergodic (irreducible, aperiodic, positive recurrent)

Let P be transition matrix of X_n

We know that $\exists\, \pi = \pi P$ $\pi_j > 0$ $\forall j$

Let $Y_n = X_{N-n}$, $n = 0, 1, \dots, N$ and consider $Y_0, Y_1, \dots, Y_N$ which is X_n run backward.

Is Y_n a Markov chain?
If yes, what is the transformation matrix?

$$P(Y_n = j \mid Y_{n-1} = i) = P(X_{N-n} = j \mid X_{N-n+1} = i)$$

$$= \frac{P(X_{N-n+1} = i \mid X_{N-n} = j)\, P(X_{N-n} = j)}{P(X_{N-n+1} = i)}$$

$$= P_{ji} \cdot \frac{\pi_j}{\pi_i} \overset{\text{def}}{=} Q_{ij}$$

Now $P(Y_0 = i_0, \dots, Y_N = i_N) = P(X_0 = i_N, \dots, X_N = i_0)$

$= \pi_{i_N} P_{i_N i_{N-1}} \cdots P_{i_1 i_0}$

$= \pi_{i_N} \cdot \left(\frac{\pi_{i_{N-1}}}{\pi_{i_N}} \cdot Q_{i_{N-1} i_N}\right) \cdots \left(\frac{\pi_{i_1}}{\pi_{i_2}} Q_{i_1 i_2}\right) = \pi_{i_1} Q_{i_1 i_2} \cdots Q_{i_{N-1} i_N}$

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so $P(Y_n = i_n \mid Y_{n-1} = i_{n-1}, \dots, Y_0 = i_0) = P(Y_n = i_n \mid Y_{n-1} = i_{n-1})$

which implies that Y_n is a Markov chain with transition probability $Q = \{Q_{ij}\}$

Definition: If $P_{ij} = \frac{\pi_j}{\pi_i} P_{ji}$ the Markov chain is time reversible.

An alternative expression is

(*) $\pi_i P_{ij} = \pi_j P_{ji}$ all i, j

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