Lemma The failure rate function $r(t)$ determines a continous distribution function having a density.

Proof: $r(t) = \frac{f(t)}{1-F(t)} = \frac{\frac{\partial}{\partial t} F(t)}{1-F(t)}$

$= -\frac{\partial}{\partial t} \cdot \log(1-F(t))$

$\Rightarrow \log(1-F(t)) = -\int_0^t r(x)dx + k$

$\Rightarrow 1-F(t) = e^k \exp(-\int_0^t r(x)dx)$

$\Rightarrow$ for $k=0$

$F(t) = 1 - \exp(-\int_0^t r(x)dx)$ ∎

Remark: Memorylessness means that $r(t)$ is constant, i.e. $r(t)=c$

But then $F(t) = 1 - e^{-ct}$

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Example. Let $X_1, \dots, X_n$ be independent

$X_i \sim \lambda_i \exp(-\lambda_i x)$ $\lambda_i \neq \lambda_j$

Let T have distribution

$P(T=j) = P_j$ and be independent of

$X_1, \dots, X_n$

Then X_T is a hyperexponential random variable.

Motivation: A population consists of n subpopulations and each subpopulation described by $X_i \sim \lambda_i \exp(-\lambda_i x)$ $\lambda_i \neq \lambda_j$

If a random individual is picked the distribution is given by T and the lifetime by $X = X_T$

Distribution of X_T ?

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$1-F(x) = \sum_{i=1}^n P(X>t \mid T_i = i) \underbrace{P(T_i = i)}_{P_j}$

$= \sum_{i=1}^n \exp(-\lambda_i t) P_j$

pdf. $f(t) = \sum_{i=1}^n \lambda_i \exp(-\lambda_i t) \cdot P_i$

hazard rate function $r(t) = \frac{\sum_{i=1}^n \lambda_i P_i \exp(-\lambda_i t)}{\sum_{i=1}^n \exp(-\lambda_i t) P_j}$ $\boxed{X = X_T}$

But $P(T=j \mid X>t) =$

$\frac{P(X>t \mid T=j) \cdot P(T=j)}{P(X>t)} =$

$(P_j \exp(-\lambda_j t) / \sum_{j=1}^n P_i \exp(-\lambda_i t))$

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so $r(t) = \sum_{j=1}^n \lambda_i P(T=j \mid X>t)$

Suppose then $\lambda_1 = \min(\lambda_i)$, i.e. $\lambda_1 < \lambda_j$ $j=2,\dots,n$

Then $P(T=1 \mid X>t) = \frac{P_1 \cdot \exp(-\lambda_1 t)}{\sum_{j=1}^n P_j e^{-\lambda_j t}}$

$= \frac{P_1}{P_1 + \sum_{j=1}^n P_j \exp(-\underbrace{(\lambda_j - \lambda_1)}_{>0})} \to 1$ when $t \to \infty$

Similarly $P(T=j \mid X>t) \to 0$ when $t \to \infty$

so $r(t) \to \lambda_1 = \min(\lambda_j)$

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gamma(n, λ) corresponds to

p.d.f. $\frac{1}{\Gamma(n)} \lambda^n x^{n-1} e^{-\lambda x}$, $n=1$ exponential dencity.

moment generating function

$\int_0^\infty \frac{1}{\Gamma(n)} \lambda^n x^{n-1} e^{-(\lambda - t)x} dx$ $t < \lambda$.

$= [\frac{\lambda}{\lambda - t}]^n = \underbrace{[\varphi(t)]^n}_{\text{m.g.f. exponential}} = \underbrace{E \exp t(X_1 + \dots + X_n)}_{\text{when } X_1, \dots, X_n \text{ are independent}}$

so if $X_1, \dots, X_n$ independent, $X_i \sim \lambda \exp(-\lambda x)$

$\Rightarrow X_1 + \dots + X_n \sim \text{gamma}(n, \lambda)$

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Lemma If $X_1 \sim \lambda_1 \exp(-\lambda_1 x)$, $X_2 \sim \lambda_2 \exp(-\lambda_2 x)$ and X_1 and X_2 are independent, then

$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Proof. $P(X_1 < X_2) = \int_0^\infty P(X_1 < X_2 \mid X_1 = x) \lambda_1 \exp(-\lambda_1 x) dx$

$= \int_0^\infty P(X_2 > x) \cdot \lambda_1 \exp(-\lambda_1 x) dx$

$= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx$

$= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \underbrace{\int_0^\infty (\lambda_1 + \lambda_2) e^{-(\lambda_1 + \lambda_2)x} dx}_{1} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

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Lemma If $X_1, \dots, X_n$ independent $X_i \sim \lambda_i \exp(-\lambda_i x)$, then $\min(X_i) \sim \lambda \exp(-\lambda x)$, $\lambda = \sum_{j=1}^{n} \lambda_j$

Proof. $P(\min X_i > x) = P(\text{all } X_i > x)$

$$= \prod_{i=1}^{n} P(X_i > x) = \prod_{i=1}^{n} e^{-\lambda_i x}$$

$$= \exp(-(\lambda_1 + \dots + \lambda_n)x)$$ ☒

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Lemma Assume $X_1, \dots, X_n$ independent $X_i \sim \lambda_i \exp(-\lambda_i x)$

Then $P(X_j = \min(X_i)) = \dfrac{\lambda_j}{\lambda_1 + \dots + \lambda_n}$

Proof.

$$P(X_j = \min X_i) = P(X_j < \min_{j \neq i} X_i)$$

But $\min_{i \neq j} X_i \sim (\sum_{j \neq i} \lambda_i) \exp(-\sum_{j \neq i} \lambda_j)$

Thus by a previous lemma

$$P(X_j < \min_{i \neq j} X_i) = \frac{\lambda_j}{\lambda_j + \sum_{i \neq j} \lambda_i} = \frac{\lambda_j}{\lambda_1 + \dots + \lambda_n}$$ ☒

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