exp.l $f(x) = \begin{cases} \lambda \exp(-\lambda t) & t \geq 0 \\ 0 & \text{else} \end{cases}$

Proposition: If $X_1, \dots, X_n$ independent, $X_i \sim \lambda_i \exp(-\lambda_i t)$

then (i) $\min X_i \sim \lambda \exp(-\lambda t)$, $\lambda = \sum_{i=1}^n \lambda_i$

(ii) $\min X_i$ and the rank order of $X_1, \dots, X_n$ are independent.

Proof (i) seen already)

(ii) Consider a particular ordering and $P(X_{i_1} < \dots < X_{i_n} \mid \min X_i) <$

But on event $\min X_i > t$ all variables are larger than $t$. By the memoryless property the remaining life times are exponentially distributed with original rates.

Hence $P(X_{i_1} - t < \dots < X_{i_n} - t \mid \min X_i) = P(X_{i_1} < \dots < X_{i_n})$ which does not depend on $\min X_i$.

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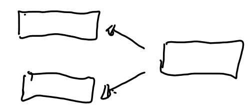
Example

(1) $\lambda_1$, $\lambda_2$



Both clerks busy, new customer assigned to first freed clerk.

$T$: time spent in office by new customer.

$E(T)$ ?

Introduce $R_1, R_2$ remaining service times

$$E[T] = E[T \mid R_1 < R_2] \cdot P(R_1 < R_2) + E[T \mid R_2 < R_1] \cdot P(R_2 < R_1)$$
$$= E[T \mid R_1 < R_2] \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + E[T \mid R_2 < R_1] \cdot \frac{\lambda_2}{\lambda_1+\lambda_2}$$

Write $T = \min(R_1, R_2) + S$, $S$ service time of new customer

$$E[T \mid R_1 < R_2] = E[R_1 \mid R_1 < R_2] + E[S \mid R_1 < R_2] = E[R_1 \mid \min R_i = R_1] + \frac{1}{\lambda_1} = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}$$

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Similarly $E[T \mid R_2 < R_1] = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}$

so $$E[T] = \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1}\right) \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + \left(\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2}\right) \cdot \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{3}{\lambda_1+\lambda_2}$$

(2) alternative solution

$T = \min(R_1, R_2) + S$

$$E[T] = \frac{1}{\lambda_1+\lambda_2} + E[S] = \frac{3}{\lambda_1+\lambda_2}$$

$$E[S] = E[S \mid R_1 < R_2] \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + E[S \mid R_2 < R_1] \cdot \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{1}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_2} \cdot \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{2}{\lambda_1+\lambda_2}$$

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Definition: Suppose $X_1, \dots, X_n$ independent $X_i \sim \lambda_i \exp(-\lambda_i t)$

Then the random variable $X_1 + \dots + X_n$ is hypoexponential.

Distribution of $X_1 + \dots + X_n$ ?

First $n = 2$ m.g.f. $X_1 + X_2$: $\frac{\lambda_1}{\lambda_1 - t} \cdot \frac{\lambda_2}{\lambda_2 - t}$, $t < \lambda_1$, $t < \lambda_2$

$$\frac{\lambda_1}{(\lambda_1-\lambda_2)} \cdot \frac{\lambda_2}{(\lambda_2 - t)} + \frac{\lambda_2 \lambda_1}{(\lambda_2-\lambda_1)(\lambda_1 - t)} = \frac{\lambda_1 \lambda_2}{(\lambda_1 - t)(\lambda_2 - t)} \left[ \underbrace{\frac{\lambda_1 - t - \lambda_2 + t}{\lambda_1 - \lambda_2}}_{=1} \right]$$

so the p.d.f of $X_1 + X_2$ is

$$\frac{\lambda_1}{\lambda_1 - \lambda_2} \cdot \lambda_2 \exp(-\lambda_2 t) + \frac{\lambda_2}{\lambda_2 - \lambda_1} \cdot \lambda_1 \exp(-\lambda_1 t) = \sum_{i=1}^{2} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \cdot \lambda_i e^{-\lambda_i t}$$

Let $$C_{i,n} = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

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Lemma If $X_1, \dots, X_n$ independent $X_i \sim \lambda_i \exp(-\lambda_i t)$

then $f_{X_1 + \dots + X_n}(x) = \sum_{i=1}^n C_{i,n} \lambda_i \exp(-\lambda_i x)$

Proof. O.K for $n = 2$.

Suppose OK for $n$. Then

$$\prod_{i=1}^{n+1} \frac{\lambda_i}{\lambda_i - t} = \left(\prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t}\right) \frac{\lambda_{n+1}}{\lambda_{n+1} - t}$$

$$\left(\sum_{i=1}^n C_{i,n} \frac{\lambda_i}{\lambda_i - t}\right) \cdot \frac{\lambda_{n+1}}{\lambda_{n+1} - t}$$

$$= \sum_{i=1}^n C_{i,n} \cdot \left[ \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \cdot \frac{\lambda_{n+1}}{\lambda_{n+1} - t} + \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_i} \cdot \frac{\lambda_i}{\lambda_i - t} \right]$$

$$= K_{n+1} \frac{\lambda_{n+1}}{\lambda_{n+1} - t} + \sum_{i=1}^n C_{i,n+1} \cdot \frac{\lambda_i}{\lambda_i - t}$$

But $$\prod_{i=1}^{n+1} \frac{\lambda_i}{\lambda_i - t} = \frac{\lambda_1}{\lambda_1 - t} \cdot \prod_{i=2}^{n+1} \frac{\lambda_i}{\lambda_i - t}$$

$$= K_1 \cdot \frac{\lambda_1}{\lambda_1 - t} + \sum_{i=2}^{n+1} C_{i,n+1} \frac{\lambda_i}{\lambda_i - t}$$

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This implies by considering the densities and multiplying with

$\exp(\min \lambda_i) \cdot t$ that

$K_{n+1} = C_{n+1,n+1}$ which concludes the proof

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