Let $f_i = P(\exists n > 0$ so that $X_n = i \mid X_0 = i)$

Recurrent stat : $f_i = 1$

Transient state $f_i < 1$

i Recurrent stat $\iff$ expectation of number returns to i is ∞.

Let $I_n = \begin{cases} 1 & X_n = i \\ 0 & \text{else} \end{cases}$

$$E\left[\sum_{n=1}^{\infty} I_n \mid X_0 = i\right] = \sum_{n=1}^{\infty} E[I_n \mid X_n = i] = \sum_{n=1}^{\infty} P(X_n = i \mid X_0 = i) = \sum_{n=1}^{\infty} P_{ii}^n$$

Proposition 4.1

State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

— " — trancient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$

Remark 4.3.1 In a finite state Markov chain all states cannot be transient. If they well all states are visited only a finite number of times which is impossible since there finite number of states

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Corollary 4.2

state i recurrent
state i communicates with state j, $j \leftrightarrow i$ $\Rightarrow$ j is recurrent

Proof.

$\exists$ k and m so that

$P_{ij}^k > 0$ $P_{ji}^m > 0$

Than $P_{jj}^{m+n+k} \geq P_{ji}^m P_{ii}^n P_{ij}^k$

probability to go from j to j in $m+n+k$ steps

probability to go from j to j in $m+n+k$ step but via state

Hence $\sum_{n=1}^{\infty} P_{jj}^{m+n+k} \geq P_{ji}^m P_{ij}^k \sum_{n=1}^{\infty} P_{ii}^n = \infty$

$P_{ji}^m \left(\sum_{n=1}^{\infty} P_{jj}^n\right) P_{ji}^k \rightarrow \sum_{n=1}^{\infty} P_{jj}^n = \infty$ i.e j recurrent

Remark 4.3.2 Trancience is a class property. If i is transient and communicates with j, j must be transient, since if it were recurrent also i would be recurrent

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Remark 4.3.3 A finite irreducible chain has one class. The states cannot be transient, therefore the states must be recurrent

Example states $\{0, 1, 2, 3\}$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}$$

Then all state communicates, e.g $0 \to 2 \to 1 \to 0$

Example, states $\{0, 1, 2, 3, 4\}$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 \\ 1 & \tfrac{1}{2} & \tfrac{1}{2} & 0 & 0 & 0 \\ 2 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ 3 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 0 \\ 4 & \tfrac{1}{4} & \tfrac{1}{4} & 0 & 0 & \tfrac{1}{2} \end{array}$$

Classes $\underbrace{\{0,1\} \ \{2,3\}}_{\text{recurrent}}$ $\underbrace{\{4\}}_{\text{transient}}$

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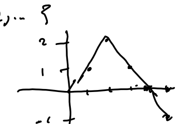
Example, random walk

state space is $\{0, \pm 1, \pm 2, \dots\}$



$P_{i,i+1} = p = 1 - P_{i,i-1}$

$0 < p < 1$

$p = 1/2$ symmetric random walk

Is the chain recurrent or trancient ?

Remark $P_{nn}^{2n-1} = 0$ $n = 1, 2, \dots$

$P_{00}^{2n} = P(X_{2n} = 0 \mid X_0 = 0) = P(n$ upward and n downward moves $\mid X_0 = 0)$

$$= \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n! \cdot n!} [p(1-p)]^n$$

Stirlings formula: $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$

when $a_n \sim b_n \iff \frac{a_n}{b_n} \to 1$ $n \to \infty$

$$P_{00}^{2n} \sim \frac{(2n)^{2n+1/2}}{n^{n+1/2} n^{n+1/2}} \cdot \frac{e^{-2n}}{e^{-n} e^{-n}} \cdot \frac{\sqrt{2\pi}}{2\pi} [4p(1-p)]^n$$

$$= \frac{2^{2n+1/2}}{\sqrt{n}} \cdot \frac{1}{\sqrt{2}\sqrt{\pi}} [p(1-p)]^n = \frac{[4 \cdot p(1-p)]^n}{\sqrt{\pi} \cdot \sqrt{n}}.$$

But $4p(1-p) \leq 1$ so $\sum_{n=1}^{\infty} \frac{[4p(1-p)]^n}{\sqrt{\pi}\sqrt{n}} = \begin{cases} \infty & \text{when } p = 1/2 \\ < \infty & \text{when } p \neq 1/2 \end{cases}$

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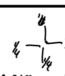
Example 2 dimencien



Symmetric Random walk in two dimencion is recurrent

In three dimension the symmetric random walk is not recurrent

4.4 Long-run proportions and limiting probabilities

X_n Markov chain

Let $f_{ij} = P(X_n = j$ some $n \mid X_0 = i)$

Proposition 4.3 If i is recurrent and $i \leftrightarrow j$ then $f_{ij} = 1$

Proof $P_{ij}^n > 0$ some n since $i \leftrightarrow j$. Fix

Define 0-1 variable : success if $X_{n_0} = j$, failure if $X_{n_0} \neq j$

$P(\text{success}) = P_{ij}^{n_0}$

Let t_1 be the first time after n_0 that X_n enters i.

Define a 0-1 variable success if $X_{t_1+n_0} = j$, failure if $X_{t_1+n_0} \neq j$

$P(\text{success}) = P_{ij}^{n_0}$

Let t_2 be the first time after t_1+n_0 that X_n enter i

and define a 0-1 variable success if $X_{t_2+n_0} = i$, failure if $X_{t_2+n_0} \neq i$

$P(\text{success}) = P_{ij}^{n_0}$

The 0-1 variables are independant since X_n starts over after visit to i. Hence a Bernoulli sequence

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Hence number of failures before first success is geometric distributed. with $p = P_{ij}^{n_0}$.

and a success will therefore occure with probability 1. ∎

Assume that j recurrent

$N_j = \min\{n > 0, X_n = j\}$ and $m_j = E[N_j \mid X_0 = j]$

Definition: A recurrent state j

is positive recurrent if $m_j < \infty$ and it

is null recurrent if $m_j = \infty$

Suppose that X_n is irreducible and recurrent

Want to show that $1/m_j$ is "the long-run proportion of the time X_n spends in j", denoted by π_j

Proposition 4.4 If the Markov chain is irreducible and recurrent then

$1/m_j = \pi_j$

whatever the inital state is.

Proof. Suppose $X_0 = i$.

Define $T_1 = \min\{n > 0, X_n = j\}$

$T_1 + T_2 = \min\{n > T_1, X_n = j\}$

$T_1 + T_2 + T_3 = \min\{n > T_1 + T_2, X_n = j\}$

T_1, T_2, T_3 is the length between visits to j

By proposition 4.3 T_1 finite

by Markov property $T_2, T_3, \dots$ i.i.d $E[T_k] = m_j$ $k = 2, 3, \dots$

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The n'th transition into j occurs at

$T_1 + \cdots + T_n$

Hence $\pi_j = \lim \frac{n}{\sum_{i=1}^{n} T_i} = \frac{1}{\lim \left[\frac{T_1}{n} + \frac{T_2 + \cdots + T_n}{n}\right]}$

But $\frac{T_1}{n} \to 0$ and by the strong law of large numbers

$$\frac{T_2 + \cdots + T_n}{n} = \left(\frac{n-1}{n}\right)\left(\frac{T_2 + \cdots + T_n}{n-1}\right) \to m_j$$

and hence $\pi_j = 1/m_j$ ∎

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