— Computation of transition matrices.

Defined $r_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_j & i = j \end{cases}$

$R = \{r_{ij}\}$, infinitesimal generator matrix

Backwards: $P'(t) = R\,P(t)$

Forwards: $P'(t) = P(t)\,R$

Solution: $P(t) = e^{Rt}$

$$e^{Rt} = \sum_{n=0}^{\infty} R^n \cdot \frac{t^n}{n!}$$

For computation this formula can be complicated, especially because of the negative diagonal elements of R.

Two alternative methods

Method 1. Based on a matrix version of

$$e^{t} = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n$$

This leads to considering

$$e^{Rt} = \lim_{n\to\infty} \left(I + R\cdot\frac{t}{n}\right)^n \simeq \left(I + R\frac{t}{n}\right)^n$$

n large.

If M one computes M, M^2, M^4 .. etc

$M = \left(I + R\cdot\frac{t}{n}\right)$

Method 2.

Use $e^{-Rt} = \lim_{n\to\infty} \left(I - R\frac{t}{n}\right)^n \approx \left(I - R\cdot\frac{t}{n}\right)^n$

n large.

Hence $P(t) = e^{Rt} \approx \left(I - R\cdot\frac{t}{n}\right)^{-n} = \left\{\left[I - R\cdot\frac{t}{n}\right]^{-1}\right\}^n$

Chapter 7. Renewal theory.

Poisson process: Times between events i.i.d exponentially distributed variables

Renewal process: Times between events i.i.d non-negative $\sim f, F$

$\{N(t), t \geq 0\}$ counting process

X_n time between $(n-1)$st event and n'th event, interarrival times.

Definition: If $X_1, X_2, ..$ are i.i.d non-negative the counting process is a renewal process

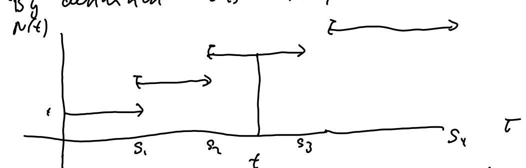
Define: $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ $n \geq 1$

Assume, $X_i \sim F$, $F(0) = P(X_i = 0) < 1$

Let $\mu = E(X_i)$

Question 1: Is an infinite number of events or renewals possible; i.e $N(t) = \infty$ for some t. ?

By definition $N(t) = \max\{n : S_n \leq t\}$.



e.g If $S_2 \leq t$, $S_3 > t$ two renewals before t and 3 after t so $N(t) = 2$.

Now, by the strong law of large numbers

$\frac{S_n}{n} \to \mu$ almost surely as $n \to \infty$

But $\mu > 0$ so $\lim_{n\to\infty} S_n = \infty$, so S_n can be less than t only a finite number of times since S_n is monotonely increasing

Hence $N(t) < \infty$. $\forall t$.

Question 2. Is $N(\infty) = \lim_{t\to\infty} N(t) = \infty$?

If $N(\infty)$, i.e the total number of renewals, is finite only a finite number of events have occurred, so then one of the interarrival times must be infinite

Then $P(N(\infty) < \infty) = P(\cup X_i = \infty)$
$\leq \sum P(X_i = \infty) = 0$

so $\lim_{t\to\infty} N(t) = \infty$

Remark 7.1.1 In renewal theory the rate at which $N(t)$ (and $E[N(t)]$) tends to ∞ is studied, i.e how many renewals there are in $(0,t)$ as $t \to \infty$.

7.2 Distribution of $N(t)$.

$N(t) \geq n \iff S_n \leq t$

$N(t) < n \iff S_n > t$

$P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n+1)$
$= P(S_n \leq t) - P(S_{n+1} \leq t)$
$= F_n(t) - F_{n+1}(t)$

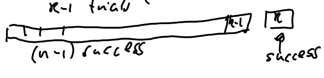
Here $S_n \sim F_n$

Example: $X_n \sim$ geometric(p). NB events/renewals will occur at integers

$P(X_n = i) = p\cdot(1-p)^{i-1}$, $i = 1, 2, ..$

$S_n = X_1 + ... + X_n$ is negative binomially distributed

$$P(S_n = k) = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & k \geq n \\ 0 & k < n \end{cases}$$



Remark that since the X_n's ~~do~~ are integer valued

$\max N(s) = [t]$; $[x]$ integer value.
$s \le t$

$P(S_n = k) = 0 \qquad k < n$

$$P(N(t)=n) = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1}(1-p)^{k-n-1}$$

Also

$$P(N(t)=n) = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

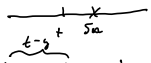
since events occur with probability p at times $1, 2, \dots, [t]$.

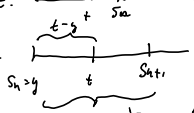
Conditioning on S_n gives an alternative expression for $P(N(t)=n)$

$$P(N(t)=n) = \int_0^\infty P(N(t)=n \mid S_n=y)\, f_{Y_n}(y)\,dy$$

$$= \int_0^t \underline{P(X_{n+1} > t-y} \mid S_n=y)\, f_{Y_n}(y)\,dy$$

Two things can happen

(i) $S_n > t$:  so $N(t) < n$, hence $\int_0^t$

(ii) $S_n \le t$  , so when $X_{n+1} > t-y$ then $N(t) = n$.

Example 7.2 $\quad X_i \sim \exp(\lambda)$

$S_n = X_1 + \dots + X_n \sim \text{gamma}(n, \lambda)$

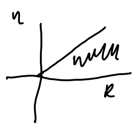
Hence $P(N(t)=n) = \int_0^t e^{-\lambda(t-y)} \cdot \underbrace{\frac{\lambda \cdot (\lambda y)^{n-1}}{(n-1)!} e^{-\lambda y}}_{f_{S_n}(y)} dy$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1}\,dy = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Poisson.

Let $m(t) = E[N(t)]$, mean-value or renewal function.

Then $m(t) = E[N(t)] = \sum_{k=1}^\infty k\, P(N(t)=k)$



$$= \sum_{k=1}^\infty \left(\sum_{n=1}^k 1\right) \cdot P(N(t)=k)$$

$$= \sum_{n=1}^\infty \sum_{k=n}^\infty P(N(t)=k)$$

$$= \sum_{n=1}^\infty P(N(t) \ge k).$$

$$= \sum_{n=1}^\infty P(S_n \le t) \quad \text{since } N(t) \ge n \Leftrightarrow S_n \le t$$

$$= \sum_{n=1}^\infty F_n(t).$$

Remark 7.2.1 The renewal function $m(t) = E[N(t)]$ uniquely defines the renewal process. This can be shown.

— " — 7.2.2. It can also be shown that $m(t) < \infty \quad \forall t < \infty$.

Remark 7.2.3 If $\{N(t), t \ge 0\}$ is a Poisson process

$P(N(t)=n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

$E[N(t)] = \lambda t$ which is linear in t.

From the previous remark, since $m(t)$ uniquely defines the renewal process no other process can have a linear mean-value or renewal function.

— " — The fact $P(N(t) < \infty) = 1$ is not sufficient for $E[N(t)] < \infty$.

This is a general result.

Integral equation for $m(t)$:

Assume that $X_n \sim_{\text{iid}} f, F$

Then $m(t) = E[N(t)] = \int_0^\infty E[N(t) \mid X_1 = x)\, f(x)\,dx$

But $E[N(t) \mid X_1 = x] = \begin{cases} 1 + E[N(t-x)] & x < t \\ 0 & x \ge t \end{cases}$

Thus $m(t) = \int_0^t [1 + m(t-x)]\, f(x)\,dx$

$= F(t) + \int_0^t m(t-x)\, f(x)\,dx.$

which is the <u>renewal equation</u>.

In some cases the renewal equation can be solved explicitly.

Example 7.3 $\qquad f(t) = I_{(0,1)}(t)$

Then $m(t) = t + \int_0^t m(t-x)\,dx = t + \int_0^t m(y)\,dy$

so $m'(t) = \underbrace{1 + m(t)}_{h(t)}$

$h'(y) = h(y)$

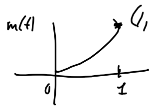
$\frac{\partial}{\partial t} \log(h(t)) = 1$

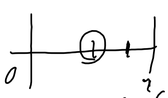
$\log(h(t)) = t + c$

$h(t) = K \cdot e^{t+c}$

so $m(t) = K \cdot e^t - 1$, since $m(0) = 0$ $K = 1$

Therefore $m(t) = e^t - 1$ for $0 \le t \le 1$.





Expected number of renewals in $(0, 1)$ is larger than 1.