UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	STK1130 — Modelling by stochastic processes
Day of examination:	Wednesday June 10 2009.
Examination hours:	14.30 – 17.30.
This examination set consists of 3 pages.	
Appendices:	None.
Permitted aids:	Accepted calculator. Formulae note for STK1100 and STK1110.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

At all time-points an urn contains N balls, some white and some black. At each time-point one ball is drawn randomly from the urn and substituted by a white ball with probability p and by a black ball with probability $1-p$. Let X_n be the number of white balls in the urn after n time units. We will in the first part of the exercise assume $0 < p < 1$.

(a) Explain why $\{X_n, n \geq 0\}$ is a time-homogeneous Markov chain.

(b) What is the state space of the Markov chain?

What are the classes?

What are their periods?

Are the classes transient, null recurrent or positive recurrent?

(c) Calculate the transition probabilities P_{ij} for all i, j.

(d) Let $\pi_j = \lim_{n\to\infty} P_{ij}^n$ where $P_{ij}^n = P(X_n = j | X_0 = i)$.

For $N = 2$, find π_j.

From the form of this solution, guess what the solution is for general N. (You do not need to make a mathematical verification of this guess.)

(Continued on page 2.)

(e) Assume now $p = 1$. What is the expected time until there are only white balls if we start with $X_0 = i$?

Problem 2.

Assume $\{X(t), t \geq 0\}$ is a Brownian motion, that is

(i) $X(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has stationary and independent increments

(iii) for each $t > 0$, $X(t)$ is normally distributed with expectation 0 and variance $\sigma^2 t$.

Let T_a be the first time-point the process hits a.

(a) Show that for $a > 0$ we have

$$P(T_a \leq t) = 2P(X(t) \geq a).$$

Problem 3.

The Bering-Chukchi-Beafort population of Bowhead whales is a threatened whale species which at spring migrates from the western and central Bering Sea up to the eastern Beaufort Sea. Estimation of the population size is performed through counting whales that pass trough a channel between the winter and summer areas.

(a) A possible model for whales passing this channel is that they follow a Poisson process with intensity λ.

You are given the following definition of the Poisson process,

(i) $\{N(t), t \geq 0\}$ is a counting process with $N(0) = 0$.

(ii) The process has stationary and independent increments.

(iii) $P(N(h) = 1) = \lambda h + o(h)$

(iv) $P(N(h) \geq 2) = o(h)$

Discuss if these assumptions are reasonable in the given case.

We will in the following assume that such a Poisson process is reasonable regardless with respect to what you have argued before.
Not all whales will be observed due to different reasons (being under water, bad sights etc). Let d be the probability for observing a whale and assume all whales are observed independently of each other.

(Continued on page 3.)

(b) If $Y(t)$ is the number of whales observed at time-point t, what kind of process will $\{Y(t), t \geq 0\}$ now be? Give a mathematical explanation for your answer.

Assume now the observations are done by human observations. If there is a long time period without observations, it is reasonable to assume that the probability for observing the whale decreases. Assume therefore that the probability for observing a whale is a decreasing function $d(s)$ where s is the time since the previous observation of a whale.

(c) Let X_i be the time between $i-1$th and ith observation.

Assume first $P(X_1 > t|N(t) = n) = \left(\frac{t-D(t)}{t}\right)^n$ where $D(t) = \int_0^t d(s)ds$.

Find $\Pr(X_1 \leq t)$.

Argue why $X_1, X_2, ...$ are independent and identically distributed.

If $Y(t)$ still is the number of observed whales at time-point t, what kind of process will then $\{Y(t), t \geq 0\}$ be? Give a reason for your answer.

(d) Show that $P(X_1 > t|N(t) = n) = \left(\frac{t-D(t)}{t}\right)^n$.

END