Exercises

*1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, $i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine **P** for this Markov chain.

*4. Consider a process $\{X_n, n = 0, 1, \ldots\}$, which takes on the values 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$$
$$= \begin{cases} P^{\mathrm{I}}_{ij}, & \text{when } n \text{ is even} \\ P^{\mathrm{II}}_{ij}, & \text{when } n \text{ is odd} \end{cases}$$

where $\sum_{j=0}^{2} P^{\mathrm{I}}_{ij} = \sum_{j=0}^{2} P^{\mathrm{II}}_{ij} = 1, i = 0, 1, 2$. Is $\{X_n, n \geqslant 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

5. A Markov chain $\{X_n, n \geqslant 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.

6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\mathbf{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix}$$

7. In Example 4.4 suppose that it has rained neither yesterday nor the day before yesterday. What is the probability that it will rain tomorrow?
8. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?
9. In a sequence of independent flips of a fair coin that comes up heads with probability .6, what is the probability that there is a run of three consecutive heads within the first 10 flips?
10. In Example 4.3, Gary is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days?
11. In Example 4.3, Gary was in a glum mood four days ago. Given that he hasn't felt cheerful in a week, what is the probability he is feeling glum today?
12. For a Markov chain $\{X_n, n \geqslant 0\}$ with transition probabilities $P_{i,j}$, consider the conditional probability that $X_n = m$ given that the chain started at time 0 in state i and has not yet entered state r by time n, where r is a specified state not equal to either i or m. We are interested in whether this conditional probability is equal to the n stage transition probability of a Markov chain whose state space does not include state r and whose transition probabilities are

$$Q_{i,j} = \frac{P_{i,j}}{1 - P_{i,r}}, \quad i, j \neq r$$

Either prove the equality

$$P\{X_n = m | X_0 = i, X_k \neq r, k = 1, \ldots, n\} = Q^n_{i,m}$$

or construct a counterexample.
13. Let $\mathbf{P}$ be the transition probability matrix of a Markov chain. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$, for all integers $n \geqslant r$.
14. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix}, \qquad \mathbf{P}_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

$$\mathbf{P}_3 = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}, \qquad \mathbf{P}_4 = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$