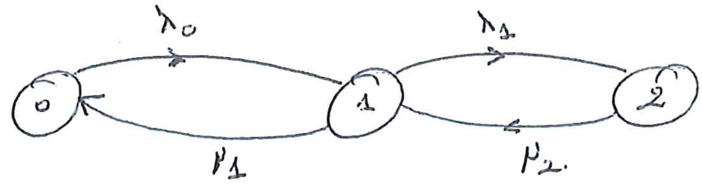
Ex 6.8



$\mu_1 = \mu_2 = \mu$
$\lambda_0 = 2\lambda$ and $\mu_n = \lambda_n = 0$ for all else.
$\lambda_1 = \lambda$

Then Thm 6.1 (or example 6.10 directly) gives us all the backward Kolmogorov equations.

For example:

$$P'_{0i}(t) = q_{01} \cdot P_{1i}(t) - \nu_0 P_{0i}(t)$$

$$\Rightarrow \quad P'_{0i}(t) = 2\lambda P_{1i}(t) - 2\lambda P_{0i}(t).$$

Ex 6.9 (with $\lambda_i = \lambda$).

Note that a pure birth process is a Poisson process, hence it is not difficult to see that, for a pure death process, we have:

$$P_{ij}(t) = \frac{1}{(i-j)!} e^{-\mu t} (\mu t)^{i-j}$$

provided $i \geq j > 0$.

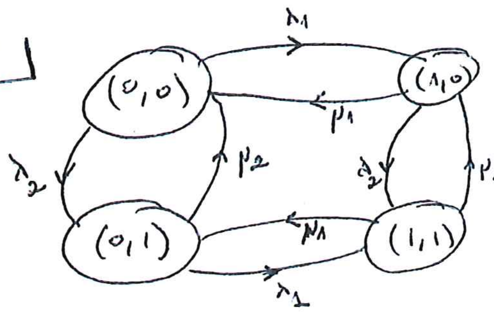
The previous reasoning doesn't apply to P_{i0}, $i \geq 1$, since then the process can become negative!

Instead we use the fact that $\sum_{j \in S} P_{ij} = 1$ to get

$$P_{i0}(t) = 1 - \sum_{k=(i-1)}^{0} \frac{1}{(i-k)!} e^{-\mu t} (\mu t)^{i-k} = 1 - \sum_{k=0}^{(i-1)} \frac{1}{k!} e^{-\mu t} (\mu t)^k$$

$$\Rightarrow \quad P_{i0}(t) = \sum_{k=i}^{\infty} \frac{1}{k!} e^{-\mu t} (\mu t)^k$$

Ex 6.10



$\nu_{00} := \nu_{(0,0)} = \lambda_1 + \lambda_2$ $\quad \lambda_{(0,0),(0,1)} = \lambda_1$
$\nu_{01} := \nu_{(0,1)} = \lambda_1 + \mu_2$ $\quad$ etc ...
$\nu_{11} := \nu_{(1,1)} = \mu_1 + \mu_2$
$\nu_{10} := \nu_{(1,0)} = \lambda_2 + \mu_1$

* Using indep., we can decompose $P_{(i,j)(k,l)}(t)$ to $P^1_{i,k}(t) \times P^2_{j,l}(t)$.

where $P^n_{i,k}(t)$ is the probability that machine $n = 1, 2$ will be in k, at time t, starting from state i.

Hence, using example 6.11, we get:

$$P^1_{0,0}(t) = \frac{1}{\lambda_1+\mu_1}\mu_1 + \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^1_{1,0}(t) = \frac{1}{\lambda_1+\mu_1}\mu_1 - \frac{\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

By symmetry:

$$P^1_{1,1}(t) = \frac{1}{\lambda_1+\mu_1}\lambda_1 + \frac{\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}$$

$$P^1_{0,1}(t) = \frac{1}{\lambda_1+\mu_1}\lambda_1 - \frac{\lambda_1}{\lambda_1+\lambda_1} e^{-(\lambda_1+\mu_1)t}.$$

Similar expression hold for $P^2_{i,j}(t)$ also and from this we can find, for instance, that:

$$P_{(0,0)(0,0)}(t) = P^1_{0,0}(t)\cdot P^2_{0,0}(t)$$

$$= \frac{1}{\lambda_1+\mu_1}\left(\mu_1+\lambda_1 e^{-(\mu_1+\lambda_1)t}\right)\cdot\frac{1}{\lambda_2+\mu_2}\left(\mu_2+\lambda_2 e^{-(\mu_2+\lambda_2)t}\right.$$

** Let's try to find the same expression "directly" from the original chain.

Kolmogorov's eq (Backward) $\Rightarrow$

$$P'_{(0,0)(0,0)}(t) = -P_{(0,0)(0,0)}(t)\cdot(\lambda_1+\lambda_2) + \lambda_2 P_{(0,1)(0,0)}(t) + \lambda_1 P_{(1,0)(0,0)}(t)$$

$$\Rightarrow P'_{(0,0)(0,0)}(t) = -P^1_{00}(t)P^2_{00}(t)(\lambda_1+\lambda_2) + \lambda_2 P^1_{00}(t)P^2_{10}(t) + \lambda_1 P^1_{10}(t)P^2_{00}(t)$$

$$= -\frac{(\lambda_1+\lambda_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\left(\mu_1+\lambda_1 e^{-(\mu_1+\lambda_1)t}\right)\left(\mu_2+\lambda_2 e^{-(\mu_2+\lambda_2)t}\right)$$

$$+ \lambda_2\frac{1}{(\lambda_1+\mu_1)}\left(\mu_1+\lambda_1 e^{-(\mu_1+\lambda_1)t}\right)\frac{1}{\lambda_2+\mu_2}\left(\mu_2-\mu_2 e^{-(\lambda_2+\mu_2)t}\right.$$

$$+ \lambda_1\frac{1}{\lambda_1+\mu_1}\left(\mu_1-\mu_1 e^{-(\lambda_1+\mu_1)t}\right)\frac{1}{\lambda_2+\mu_2}\left(\mu_2+\lambda_2 e^{-(\lambda_2+\mu_2)t}\right.$$

$$= P^1_{00}(t)\left(-\frac{(\lambda_1+\lambda_2)}{(\lambda_2+\mu_2)}\mu_2 - \frac{(\lambda_1+\lambda_2)\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t} + \frac{\lambda_2}{\lambda_2+\mu_2}\mu_2 - \frac{\mu_2\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ \lambda_1 P^1_{10}(t)P^2_{00}(t).$$

$$= P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_1+\mu_2} - \frac{e^{-(\lambda_2+\mu_2)t}}{}\left((\lambda_1+\lambda_2)\lambda_2+\mu_2\lambda_2\right)\right)+\lambda_1 P^1_{10}(t)P^2_{00}(t)$$

$$= P^1_{00}(t)\left(-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right) + P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2} - \frac{e^{-(\lambda_2+\mu_2)t}}{(\lambda_2+\mu_2)}\left((\lambda_1+\lambda_2)\lambda_2+\mu_2\lambda_2-\lambda_2(\lambda_2+\right.\right.$$

$$+\lambda_1 P^1_{10}(t)\, P^2_{00}(t).$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^1_{00}(t)\left(-\frac{\lambda_1\mu_2}{\lambda_2+\mu_2} - \frac{\lambda_1\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right)$$

$$+ P^2_{00}(t)\,\lambda_1\, P^1_{10}(t).$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\left(\frac{\lambda_1\mu_1}{\lambda_1+\mu_1} - \frac{\lambda_1\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right.$$

$$\left. - \frac{\lambda_1\mu_1}{\lambda_1+\mu_1} - \frac{\lambda_1\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right).$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\cdot \underbrace{\left(-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right)}_{=\frac{d}{dt}P^1_{00}}.$$

$$= P^1_{00}(t)\left(\frac{d}{dt}P^2_{00}(t)\right) + P^2_{00}(t)\left(\frac{d}{dt}P^1_{00}(t)\right).$$

$$= \frac{d}{dt}\left(P^1_{00}(t)\, P^2_{00}(t)\right) = \frac{d}{dt}P^{(t)}_{(0,0)(0,0)}$$

Thus the backward Kolmogorov equation holds in this case.

Similarly the forward Kolmogorov eq can be checked.