$\underline{Ex\ 10.1}$: Let $s \le t$, then we have that:

$$B(s) + B(t) = B(t) - B(s) + B(s) + B(s) - \overbrace{B(0)}^{=0} - \overbrace{B(0)}^{=0}$$
$$= B(t) - B(s) + 2(B(s) - B(0))$$

By the independence of increments property, we have: $[B(t) - B(s)] \perp\!\!\!\perp [B(s) - B(0)]$

Moreover, we know: $\begin{cases} B(t) - B(s) \sim N(0, t-s) \\ B(s) - B(0) \sim N(0, s) \end{cases}$

Thus by the "linear combination" property of Gaussian random variables, we get:

$$B(s) + B(t) \sim N(0, \underbrace{t - s + 2^2 s}_{= t + 3s})$$

$\underline{Ex\ 10.9}$: Let $s<t$, then we know that $\begin{cases} X(s) \sim N(\mu s, \sigma^2 s) \\ X(t) \sim N(\mu t, \sigma^2 t) \end{cases}$

Hence $\begin{bmatrix} X(s) \\ X(t) \end{bmatrix} \sim N_2(\Lambda, \Sigma)$ where:

$$\Lambda = \begin{bmatrix} \mu s \\ \mu t \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma^2 s & Cov(X(s), X(t)) \\ Cov(X(s), X(t)) & \sigma^2 t \end{bmatrix}$$

Now

$$Cov(X(s), X(t)) \; (= E[(X(s) - \mu s)(X(t) - \mu t)])$$
$$= E[X(s) X(t)] - E[X(s)] . E[X(t)]$$
$$= E[X(s) . (X(t) - X(s) + X(s))] - \mu s . \mu . t$$
$$= E[X(s) . (X(t) - X(s))] + E[X(s)^2] - \mu^2 s . t$$
$$(\text{indep. of increments}) = \underbrace{E[X(s)]}_{} . \underbrace{E[(X(t) - X(s))]}_{} + \overbrace{\sigma^2 . s + (\mu s)^2} - \mu^2 s . t$$
$$= \mu s \times (t-s) . \mu + \sigma^2 s + \mu^2 s^2 - \mu^2 s . t$$
$$= \cancel{\mu^2 s . t} - s^2 \mu^2 + \sigma^2 s + \mu^2 s^2 - \cancel{\mu^2 s t} = \sigma^2 s$$

Hence

$$\Sigma = \begin{bmatrix} \sigma^2 s & \sigma^2 s \\ \sigma^2 s & \sigma^2 t \end{bmatrix} \quad \square$$

Problem 4 in Exam 2007 :

a) See previous exercise 10.9 .

b) . Let $Z(t) := \frac{W(t)}{\sqrt{t}}$ then $Var(Z(t)) = \left(\frac{1}{\sqrt{t}}\right)^2 \cdot Var(W(t)) = \frac{1}{t}\sigma^2 t = \sigma^2$

. $Cov(Z(s), Z(t)) = Cov\left(\frac{W(s)}{\sqrt{s}}, \frac{W(t)}{\sqrt{t}}\right) = \frac{1}{\sqrt{s}\sqrt{t}} Cov(W(s), W(t))$

$= \frac{1}{\sqrt{s}\sqrt{t}} \cdot \sigma^2 s = \sigma^2 \cdot \frac{\sqrt{s}}{\sqrt{t}}$

Thus $corr(Z(s), Z(t)) = \dfrac{\sigma^2 \frac{\sqrt{s}}{\sqrt{t}}}{\sqrt{\sigma^2} \cdot \sqrt{\sigma^2}} = \sqrt{\frac{s}{t}}$ □

Ex 7.2 : We know that $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$ $X_i \sim Poisson(\mu)$

a) Since the interarrival times X_i's are independent, we have : $S_n \sim Poisson(\mu n)$

b) We know that (See Slide 5/19 in the slides for chapters 7-10) :

$P(N(t) = n) = F_n(t) - F_{n+1}(t)$, where F_n is CDF of S_n .

Thus

$$P(N(t) = n) = \sum_{k=0}^{\lfloor t \rfloor} \frac{e^{-n\mu}}{k!}(n\mu)^k - \sum_{k=0}^{\lfloor t \rfloor} \frac{e^{-(n+1)}}{k!}((n+1)\mu)^k .$$

Extra exercise (Ex 7.3 [ROSS 2014]) :

a) $P(N(t) = n \mid S_n = y)$? This is the probability ~~that~~, given that at time y the n-th event has happened, that by time t there is still only n events that have happened. This probability is zero if $y > t$ and is equal to $P(X_{n+1} > t-y)$ i.e. the $(n+1)$th event takes longer than $(t-y)$ amount of time to happen. Thus :

$$P(N(t) = n \mid S_n = y) = P(X_{n+1} > t-y) \cdot \mathbb{1}_{\{y-t \leq 0\}} = (1 - F(t-y)) \cdot \mathbb{1}_{\{y \leq t\}}$$

b) $$P(N(t) = n) = \int_0^\infty P(N(t) = n \mid S_n = y) f_{S_n}(y)\, dy = \int_0^\infty (1 - F(t-y)) \cdot \mathbb{1}_{\{y \leq t\}} \cdot f_{S_n}(y)\, dy$$

$$= \int_0^t (1 - (1 - e^{-\lambda(t-y)})) \cdot \frac{\lambda e^{-\lambda y}(\lambda y)^{n-1}}{(n-1)!} dy = \dots = \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t y^{n-1} dy = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Thus $N(t) \sim Poisson(\mu t)$ □