Exercise sheet 1 :

Exercise 4.2 : This is a continuation Example 4.4. and is solved in in the same way. The state space will have $2\times2\times2=8$ states.

Exercise 4.5 : We know that $E[X_3]=\sum_{x\in S} x\cdot P_{X_3}(x)$, where $S=\{1,2,3\}$ and $P_{X_3}(\cdot)$ is the probability distribution function of X_3 i.e the Markov chain after 3 steps and starting from the initial distribution $(P_{X_0}(0), P_{X_0}(1), P_{X_0}(2)) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

It remains to find $P_{X_3}(\cdot)$. By using the remark on page 204 [this is based on the fact that $P(X_n=j)=\sum_{i\in S} P(X_n=j \mid X_0=i)\,p(X_0=i)$]

we have that
$$\begin{bmatrix} P_{X_3}(0) \\ P_{X_3}(1) \\ P_{X_3}(2) \end{bmatrix}^t = \begin{bmatrix} P_{X_0}(0) \\ P_{X_0}(1) \\ P_{X_0}(2) \end{bmatrix}^t \circ P^{(3)}$$
where $P^{(3)}$ is the transition matrix of the Markov chain.

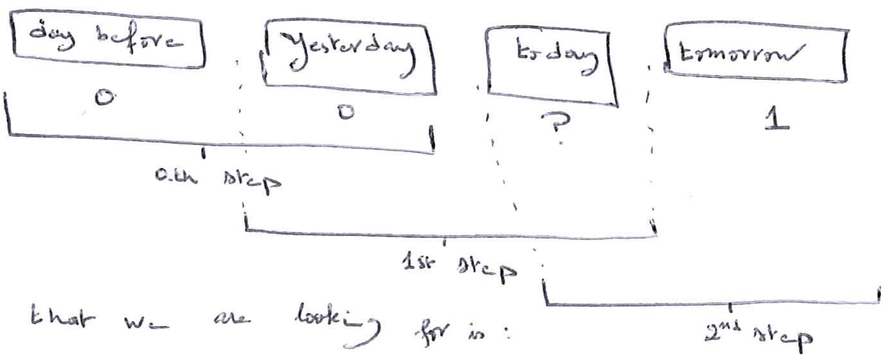
Hence :
$$\begin{bmatrix} P_{X_3}(0) \\ P_{X_3}(1) \\ P_{X_3}(2) \end{bmatrix}^t = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}^t \circ \underbrace{\begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix}}_{P^{(3)}} = \cancel{\begin{bmatrix} \frac{155}{432} \\ \frac{19}{54} \\ \frac{49}{144} \end{bmatrix}} \begin{bmatrix} \frac{59}{144} \\ \frac{43}{216} \\ \frac{169}{432} \end{bmatrix}^t$$

This gives
$$E[X_3] = 2\times\frac{169}{432} + 1\times\frac{43}{216} + 0\times\frac{59}{144}$$
$$\simeq \cancel{1.0324}\ 0.9815$$

Exercise 4.7

0 : no rain

1 : rain



The probability that we are looking for is :
$$P(\{X_2=0\} \text{ or } \{X_2=1\} \mid X_0=3) = P(X_2=0\mid X_0=3)+P(X_2=1\mid X_0=3)$$
$$= P^2_{30} + P^2_{31} = \dots = 0.26$$

Exercise 4.10 We can use the method on page 200 [Ross 2019]. To this end let $A=\{G\}$, then $P(X_3\neq G, X_2\neq G, X_1\neq G \mid X_0=C) = 1-\beta$ where $\beta = P(X_k\in A \text{ for some } k=1,2,3 \mid X_0=C)$. Using the mentioned method we find that $\beta = Q^{(3)}_{CG}$ where $Q=\begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$

This gives a final value of 0.585.

7 days. Today

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

Exercise 4.11:

$$P(X_4 = G \mid X_0 = G, X_3 \neq C, X_2 \neq C, \dots, X_{-3} \neq C) =$$

$$\boxed{P(A \mid B, C) = \frac{P(A; B \mid C)}{P(B \mid C)}}$$

$$P(\underbrace{X_4 = G}_{A} \mid \underbrace{X_3 \neq C, \dots, X_1 \neq C}_{B}, \underbrace{X_0 = G, X_{-1} \neq C, \dots, X_{-3} \neq C}_{C})$$

$$= \frac{P(X_4 = G, X_3 \neq C, \dots, X_1 \neq C \mid X_0 = G, X_{-1} \neq C, \dots, X_{-3} \neq C)}{P(X_3 \neq C, \dots, X_1 \neq C \mid X_0 = G, X_{-1} \neq C, \dots, X_{-3} \neq C)}$$

(Markov property)

$$= \frac{P(X_4 = G, X_3 \neq C, \dots, X_1 \neq C \mid X_0 = G)}{P(X_3 \neq C, \dots, X_1 \neq C \mid X_0 = G)}$$

Now we use the previous exercise and the techniques of the absorbing state A starting on page 200 [Ross 2019] to find that the above is:

$$\frac{Q^{(4)}_{GG}}{1 - Q^{(3)}_{GC}} \simeq 0.409.$$

Exercise 4.12: Recall the set up for Ex 4.10.

$$\begin{array}{c|c} & \begin{array}{ccc} C & S & G \end{array} \\ \begin{array}{c} C \\ S \\ G \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \end{array}$$

Let's test the claim for:

$$(*) \quad P(X_4 = S \mid X_3 \neq G, \dots, X_1 \neq G, X_0 = C).$$

Using the technique of "absorbing state A", this is equall to

$$\overline{Q}^{(4)}_{CS} \quad \text{where} \quad \overline{Q}^{(4)} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} 0.234 & 0.234 & 0.532 \\ 0.175 & 0.175 & 0.649 \\ 0 & 0 & 1 \end{bmatrix}$$

thus $\overline{Q}^{(4)}_{CS} = 0.234$

On the other hand and using the method proposed in the exercise, we have that the probability we are looking for is:

$$Q^{(4)}_{CS} \quad \text{where} \quad Q^{(4)} = \begin{bmatrix} \frac{0.5}{1-0.1} & \frac{0.4}{1-0.1} \\ \frac{0.3}{1-0.3} & \frac{0.4}{1-0.3} \end{bmatrix}^4 = \begin{array}{c|c} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & \begin{bmatrix} 0.49 & 0.51 \\ 0.49 & 0.51 \end{bmatrix} \end{array}$$

$$Q^{(4)}_{CS} = 0.51$$

and to find a counter example to the claim we use the same steps as in Ex 4.11, this means:

$P(A|BC) = \frac{P(A,B|C)}{P(B|C)}$

$$P(\underbrace{X_4 = S}_{A} \mid \underbrace{X_3 \neq 6 \text{ --- } X_1 \neq 6}_{b}, \underbrace{X_0 = C}_{c})$$

$$= \frac{P(X_4 = S, X_3, X_2, X_1 \neq 6 \mid X_0 = C)}{P(X_3, X_2, X_1 \neq 6 \mid X_0 = C)}$$

$$= \frac{\text{———}}{P(X_3 = C, X_2, X_1 \neq 6 \mid X_0 = C) + P(X_3 = S, X_2, X_1 \neq 6 \mid X_0 = C)}$$

$$= \frac{\overline{Q}^{(4)}_{CS}}{\overline{Q}^{(3)}_{CC} + \overline{Q}^{(3)}_{CS}} = \frac{\overline{Q}^{(4)}_{CS}}{1 - \overline{Q}^{(3)}_{C6}}$$

$$\stackrel{?}{=} Q^{(4)}_{CS}$$

Now :

$$\overline{Q}^{(3)} = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.22 & 0.561 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \overline{Q}^{(4)} = \begin{bmatrix} 0.2341 & 0.234 & 0.5319 \\ 0.1755 & 0.1756 & 0.6489 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{\overline{Q}^{(4)}_{CS}}{1 - \overline{Q}^{(3)}_{C6}} = \frac{0.234}{1 - 0.415} = 0.4$$

$$Q^4 = \begin{bmatrix} 0.491 & 0.509 \\ 0.49 & 0.51 \end{bmatrix} \Rightarrow Q^{(4)}_{CS} = 0.51 \neq 0.4$$

and thus our counter example which shows that the equality doesn't hold in general.