Exercise set 2:

Def: Let $f_i := P(\exists n > 0 \text{ s.t } X_n = i \mid X_0 = i)$, then:

A state i is recurrent if $f_i = 1$.

A state i is transient if $f_i < 1$.

Prop: State i is recurrent if $\sum_{n=1}^{\infty} P_{ii}^n = \infty$

State i is transient if $\sum_{n=1}^{\infty} P_{ii}^n < \infty$.

$\boxed{\text{Ex } 4.14}$

$$* \quad P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \xrightarrow[\text{decomp}]{\text{Eigenvalue}} P_1 = A D A^{-1}, \text{ where}$$

$$D = \text{diag}\left[-\frac{1}{2} \quad -\frac{1}{2} \quad 1\right] \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Then
$$P_1^n = A D^n A^{-1} = A \begin{bmatrix} \frac{1}{2^n} & 0 & 0 \\ 0 & \frac{-1}{2^n} & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1}$$

$$= \frac{1}{3} \begin{bmatrix} \frac{2^n - 2}{2^n} & \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} \\ \frac{2^n + 2}{2^n} & \frac{2^n - 2}{2^n} & \frac{2^n + 2}{2^n} \\ \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} & \frac{2^n + 2}{2^n} \end{bmatrix}.$$

But $\sum_{n \geq 1} \frac{2^n - 2}{2^n} = \sum_{n \geq 1}\left(1 - \frac{2}{2^n}\right) = \infty$. Also $\sum_{n \geq 1} \frac{2^n + 2}{2^n} = \infty$.

Hence all states are recurrent since all states communicate.

- $P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ Let's find the classes first this time:

- $1 \rightsquigarrow 4$.
- $2 \rightsquigarrow 4$
- $3 \rightsquigarrow 1$; $3 \rightsquigarrow 2$
- $4 \rightsquigarrow 3$

$\Rightarrow$ $\begin{cases} 4 \rightsquigarrow 3 \rightsquigarrow 2 \Rightarrow 4 \leftrightsquigarrow 2. \\ 4 \rightsquigarrow 3 \rightsquigarrow 1 \Rightarrow 4 \leftrightsquigarrow 1. \\ 3 \rightsquigarrow 1 \rightsquigarrow 4 \Rightarrow 3 \leftrightsquigarrow 4. \end{cases}$ $\Rightarrow 1 \leftrightsquigarrow 2 \leftrightsquigarrow 3 \leftrightsquigarrow 4$.

Since transience / recurrence is a class property, we now need only study one state. Taking $i = 2$ for example we find that all states are recurrent because the chain is finite. [General fact: An irreducible finite Markov chain is recurrent]

- $P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

- $4 \leftrightsquigarrow 5$
- $1 \rightsquigarrow 3$
- $2 \rightsquigarrow 1$; $2 \rightsquigarrow 3$
- $3 \rightsquigarrow 1$

$\rightarrow$ $\begin{cases} \cdot\ 4 \leftrightsquigarrow 5 \\ \cdot\ 1 \leftrightsquigarrow 3 \\ \cdot\ 2 \end{cases}$

Thus: $\{1, 3\}$, $\{4, 5\}$, $\{2\}$.

From P_3 we can see that $\{2\}$ is transient while the other two classes are recurrent.

Ex 4.20 $\quad \{X_i\}_{i \in \mathbb{N}} \subseteq \{0, \dots, M\}$.

X_i is irreducible & aperiodic $\Rightarrow$ Ergodic $\Rightarrow \exists!$ stationary distribution

unique

$\Rightarrow \exists!\ \pi = (\pi_0, \dots, \pi_M)^t$ s.t $\pi = \pi P$ i.e $\pi_j = \sum_{i=0}^{M} \pi_i P_{ij}$

But $\displaystyle\sum_{i=0}^{M} \frac{1}{M+1} P_{ij} = \frac{1}{M+1} \underbrace{\sum_{i=0}^{M} P_{ij}}_{\text{Doubly stochastic}} = \frac{1}{M+1}$

Hence $\pi = \left(\frac{1}{M+1}, \dots, \frac{1}{M+1} \right)^t$ are the long-run proportions.

4.23 X_i is a MC with values in $S = \{G, B\}$ and index set $\{0, 1, \dots\}$ representing the years ahead. The transition matrix is:

$$\begin{array}{c|cc} & G & B \\ \hline G & 0.5 & 0.5 \\ B & \frac{1}{3} & \frac{2}{3} \end{array}$$

Moreover: * # of storms is $\sim P(\lambda)$.

* λ depends on $X_i = G$ or $X_i = B$

s.t. $\begin{cases} \lambda = 1 & , X_i = G \\ \lambda = 3 & , X_i = B \end{cases}$

* $X_0 = G$

a) Let N_i denote # of storms in year i. Then:

$$E[N_1 + N_2] = E[E[N_1 \mid X_1]] + E[E[N_2 \mid X_2]]$$

$$E[E[N_1 \mid X_1]] = \sum_{j \in S} E[N_1 \mid X_1 = j] \cdot P(X_1 = j)$$

$$= \sum_{j \in S} E[N_1 \mid X_1 = j] \cdot \left(\sum_{i \in S} P(X_1 = j \mid X_0 = i) P(X_0 = i) \right)$$

$$= \sum_i \sum_j E[N_1 \mid X_1 = j] \, P(X_1 = j \mid X_0 = i) \, P(X_0 = i)$$

$$= \sum_j E[N_1 \mid X_1 = j] \, P(X_1 = j \mid X_0 = G) \underbrace{P(X_0 = G)}_{= 1}$$

$$= E[N_1 \mid X_1 = G] \cdot P_{GG} + E[N_1 \mid X_1 = B] \, P_{GB}$$

$$= 1 \times \frac{1}{2} + 3 \times \frac{1}{2} = 4 \times \frac{1}{2} = 2$$

$$E[E[N_2 \mid X_2]] = E[N_2 \mid X_2 = G] \cdot P^2_{GG} + E[N_2 \mid X_2 = B] \, P^2_{GB}$$

$$= 1 \cdot P^2_{GG} + 3 \cdot P^2_{GB} = 13/6 \quad , \text{ since}$$

$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}^2 = \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix}.$$

Hence: $E[N_1 + N_2] = 2 + \frac{13}{6} = \frac{25}{6}$.

b). $P(N_3=0) = \sum_{j\in S} P(N_3=0 \mid X_3=j)\, P(X_3=j)$.

$$= \sum_{i,j\in S} P(N_3=0\mid X_3=j)\, P(X_3=j\mid X_0=i)\, P(X_0=i)$$

$$= P(N_3=0\mid X_3=G)\, P(X_3=G\mid X_0=G) + P(N_3=0\mid X_3=B)\, P(X_3=B\mid X_0=G)$$

$$= P(N_3=0\mid X_3=G)\, P^3_{GG} + P(N_3=0\mid X_3=B)\, P^3_{GB}$$

Since $\begin{cases} P(N_3=0\mid X_3=G) = P(N_3=0\mid X=1) = \dfrac{1^0 e^{-1}}{0!} \\ P(N_3=0\mid X_3=B) = \dfrac{3^0 e^{-3}}{0!} \end{cases}$

We get: $P(N_3=0) = e^{-1} P^3_{GG} + e^{-3} P^3_{GB} = 0.17$.

c) "$E[N_\infty]$" $= E[N_\infty\mid X_\infty=G]\cdot P(X_\infty=G) + E[N_\infty\mid X_\infty=B]\, P(X_\infty=B)$

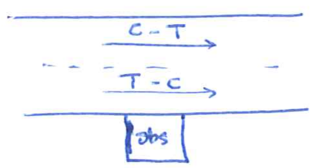
$$= E[N_\infty\mid X_\infty=G]\cdot \pi_G + E[N_\infty\mid X_\infty=B]\cdot \pi_B$$

Let's find $\pi=(\pi_G, \pi_B)$: $\pi = \pi \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ & $\pi_G + \pi_B = 1$

$$\Rightarrow \pi_G\cdot\frac{1}{2} + \pi_B\cdot\frac{1}{3} = \pi_G \Rightarrow \pi_B = \frac{3}{2}\pi_G \Rightarrow \begin{cases}\pi_G = \frac{2}{5}\\ \pi_B = \frac{3}{5}\end{cases}$$

Thus "$E[N_\infty]$" $= 1\cdot\frac{2}{5} + 3\cdot\frac{3}{5} = \frac{11}{5}$.

4.30| We can construct a Markov chain that represents the state of what an "observer is observing".



Then we have the following transition matrix:

$$P = \begin{array}{c c} & \begin{array}{cc} C & T \end{array} \\ \begin{array}{c} C \\ T \end{array} & \begin{bmatrix} \frac{4}{5} & \boxed{\frac{1}{5}} \\ \boxed{\frac{3}{4}} & \frac{1}{4} \end{bmatrix} \end{array}.$$

To solve our problem, we find $\pi = \pi P$, i.e.:

$$\begin{cases} \pi_C + \pi_T = 1. \\ \pi_T = \pi_C P_{CT} + \pi_T P_{TT} \end{cases} \Rightarrow \begin{cases} \pi_C + \pi_T = 1 \\ \pi_T = \pi_C \frac{1}{5} + \pi_T\cdot\frac{1}{4} \end{cases}$$

$$\Rightarrow \begin{cases} \pi_C + \pi_T = 1 \\ \pi_T = \pi_C \cancel{\frac{16}{15}}\ \frac{4}{15} \end{cases} \Rightarrow \begin{cases} \pi_C = \frac{15}{19} \\ \pi_T = \frac{4}{19} \end{cases}$$