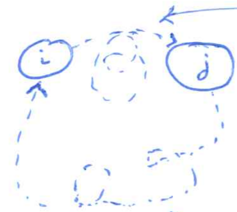
Exercise 4.16 | Let's show that if i is a recurrent state and j is another state such that $i \not\leftrightarrow j$ then $P_{ij} = 0$

$$i \not\leftrightarrow j \iff (i \to j) \text{ xor } (j \to i)$$

where xor is the exclusive or operation.

Suppose that $P_{ij} \neq 0$ i.e $P_{ij} > 0$

* If $i \to j$ then since i is recurrent we have the following picture



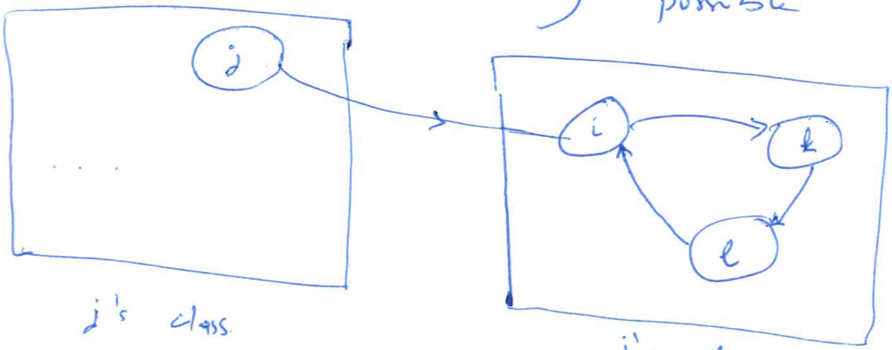
This due to $i \to j$

This is due to the recurrence of i

Thus $i \leftrightarrow j \Rightarrow$ Contradiction !!

* If $j \to i$, then since $P_{ij} > 0$ ~~to~~ we have $j \leftrightarrow i$ which leads to another contradiction.

Conclusion: If i is a recurrent state and j is in another class, then the only possible configuration is



j's class.

i's class.

Remark: This exercise explains why we call recurrent classes closed. This is also "why" we have the following theorem:

Theorem: If the state space is finite, then every closed class is recurrent.

4.35

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{array}$$

Limiting probabilities are solutions to $\pi P = \pi$.

4.52

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.6 & 0.4 \\ B & 0.3 & 0.7 \end{array}$$

Let $(\text{profit})_n$ denote the profit the driver makes at step (ride) number n.
Then the quantity of interest is $E[(\text{profit})_n]$ when n goes to ∞.

$$E[(\text{profit})_n] = E\Big[E[(\text{profit})_n \mid (\text{ride})_n]\Big]$$

$\Big((\text{ride})_n := (X_n, X_{n-1})\Big)$ random variables

$$= E\Big[E[(\text{profit})_n \mid (X_n, X_{n-1})]\Big]$$

$$= \sum_{i,j} E[(\text{profit})_n \mid (X_n, X_{n-1}) = (i,j)] \cdot P\big((X_n, X_{n-1}) = (i,j)\big)$$

$$= \sum_{i,j} E[(\text{profit})_n \mid (X_n, X_{n-1}) = (i,j)] \cdot P(X_n = i \mid X_{n-1} = j)\, P(X_{n-1} = j)$$

$$= E[(\text{profit})_n \mid (X_n, X_{n-1}) = (A,A)] \cdot P_{AA} \cdot P(X_{n-1} = A)$$
$$+ E[(\text{profit})_n \mid (X_n, X_{n-1}) = (A,B)] \cdot P_{BA} \cdot P(X_{n-1} = B)$$
$$+ E[(\text{profit})_n \mid (X_n, X_{n-1}) = (B,A)] \cdot P_{AB} \cdot P(X_{n-1} = A)$$
$$+ E[(\text{profit})_n \mid (X_n, X_{n-1}) = (B,B)] \cdot P_{BB} \cdot P(X_{n-1} = B)$$

$$= 6 \cdot P_{AA} \cdot P(X_{n-1} = A) + 12\, P_{BA} \cdot P(X_{n-1} = B) + 12\, P_{AB} \cdot P(X_{n-1} = A) + 8 \cdot P_{BB} \cdot P(X_{n-1} = B)$$

Thus: $E[\text{profit}] := \lim_{n \to \infty} E[(\text{profit})_n]$

$$= 6 \cdot P_{AA} \cdot P(X_\infty = A) + 12 \cdot P_{BA}\, P(X_\infty = B) + 12\, P_{AB} \cdot P(X_\infty = A) + 8\, P_{BB}\, P(X_\infty = B)$$

$$= 6\, P_{AA}\, \pi_A + 12\, P_{BA}\, \pi_B + 12\, P_{AB}\, \pi_A + 8\, P_{BB} \cdot \pi_B$$

where π are the long-time (stationary) distributions (sol of $\pi P = \pi$).

$\boxed{\text{Ex } 4.60}$

a) Let $f_i(n) = P(X_1 \neq 3,4 ; \ldots ; X_{n-1} \neq 3,4 ; X_n = 3 \mid X_0 = i)$

then we are seeking $f_1 := \sum_{n=1}^{\infty} f_1(n)$

Now $f_i(n) = \sum_{j \neq 3,4} P(X_2 \neq 3,4 ; \ldots ; X_n = 3 \mid X_1 = j, X_0 = i) P_{ij}$

$= \sum_{j \neq 3,4} P(X_2 \neq 3,4 ; \ldots ; X_n = 3 \mid X_1 = j) P_{ij}$ (Markov ppty)

$= \sum_{j \neq 3,4} P(X_1 \neq 3,4 ; \ldots ; X_{n-1} = 3 \mid X_0 = j) P_{ij}$ (Time homogen)

$= \sum_{j \neq 3,4} P_{ij} f_j(n-1)$

Now: $f_i = f_i(1) + \sum_{n \geq 2} f_i(n)$

$= f_i(1) + \sum_{n \geq 2} \sum_{j \neq 3,4} P_{ij} f_j(n-1)$

$= f_i(1) + \sum_{j \neq 3,4} P_{ij} \sum_{n \geq 2} f_j(n-1)$

$= f_i(1) + \sum_{j \neq 3,4} P_{ij} \underbrace{\sum_{n \geq 1} f_j(n)}_{f_j}$

$= f_i(1) + P_{i1} f_1 + P_{i2} f_2.$

<u>Thus:</u> $f_i = f_i(1) + P_{i1} f_1 + P_{i2} f_2. \quad i = 1, 2.$

i.e $\begin{cases} f_1 = f_1(1) + P_{11} f_1 + P_{12} f_2 \\ f_2 = f_2(1) + P_{21} f_1 + P_{22} f_2 \end{cases}$

with $f_1(1) = P_{13}$ & $f_2(1) = P_{23}$

hence $\begin{cases} f_1 = P_{13} + P_{11} f_1 + P_{12} f_2 \\ f_2 = P_{23} + P_{21} f_1 + P_{22} f_2 \end{cases} \Longrightarrow \begin{cases} \boxed{f_1 = \frac{11}{21}} \\ f_2 = \frac{8}{21} \end{cases}$

Note that $f_3 = 1$ and $f_4 = 0$ and hence we can write our system as follows:

$$\begin{cases} f_1 = P_{11} f_1 + P_{12} f_2 + P_{13} f_3 + P_{14} f_4 \\ f_2 = P_{21} f_1 + P_{22} f_2 + P_{23} f_3 + P_{24} f_4 \\ f_3 = 0 f_1 + 0 . f_2 + 1 . f_3 + 0 . f_4 \\ f_4 = 0 . f_1 + 0 . f_2 + 0 . f_3 + 1 . f_4 \end{cases}$$

Thus $f = [f_1, \dots, f_4]^t$ is the solution to $f = \tilde{P} f$ where

$$\tilde{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With the condition:

$$f = [f_1, f_2, \underset{\substack{\uparrow \\ \text{reach} \\ 3 \text{ first}}}{1}, 0]^t$$

b). Let $T := \inf \{ n \geq 0 : X_n = 3 \text{ or } 4 \}$ = first time to hit 3 or 4.

Then: $m_i := E_i[T] = E[T \mid X_0 = i]$

and $m_i = E_i[E[T \mid X_1]] = E[T \mid X_1 = 1] P_{i1} + E[T \mid X_1 = 2] . P_{i2}$
$+ E[T \mid X_1 = 3] . P_{i3} + E[T \mid X_1 = 4] . P_{i4}$

but $E[T \mid X_1 = i] = 1 + m_i$

Thus $m_i = 1 + m_1 P_{i1} + m_2 P_{i2} + \cancel{m_3 P_{i3}} + \cancel{m_4 P_{i4}}$

$$\Rightarrow \begin{cases} m_1 = 1 + P_{11} m_1 + P_{12} m_2 \\ m_2 = 1 + P_{21} m_1 + P_{22} m_2 \end{cases}$$

$$\Rightarrow \dots \Rightarrow \boxed{m_1 = \frac{55}{21}}$$