$\boxed{4.64}$

- $E[\#\text{ of individuals to ever exist}] = E\left[\sum_{n\geq 0} X_n\right]$

$$= \sum_{n\geq 0} E[X_n] \qquad (\text{Independence from generation to the next})$$

$$= \sum_{n\geq 0} \mu^n = \frac{1}{1-\mu} \qquad (|\mu| < 1)$$

- $E\left[\sum_{n\geq 0} X_n \,\middle|\, X_0 = n\right] = n\, E\left[\sum_{n\geq 0} X_n \,\middle|\, X_0 = 1\right] \quad (\text{Indep.})$

$$= \frac{n}{1-\mu}$$

$\boxed{4.65}$

- For $n = 0$: $\quad P(X_0 = 0 \mid X_0 = 1) = 0 \leq \pi \quad \checkmark$

- Assume $\quad P(X_{n-1} = 0) \leq \pi$ where π is any sol. of $\pi = \sum_{j\geq 0} \pi^j P_j$

Then:

$$P(X_n = 0) = \sum_{j\geq 0} P(X_n = 0 \mid X_1 = j)\, P(X_1 = j)$$

$$= \sum_{j\geq 0} \left[P(X_n = 0 \mid X_1 = 1)\right]^j P_j$$

$$\Rightarrow \quad P(X_n = 0) = \sum_{j\geq 0} P(X_{n-1} = 0 \mid X_0 = 1)^j P_j \leq \sum_{j\geq 0} \pi^j P_j = \pi$$

$$\Rightarrow \quad P(X_n = 0) \leq \pi. \quad \checkmark$$

Hence by the induction principle, for every n and π solving $\pi = \sum_{j\geq 0} \pi^j P_j$

$$\pi \geq P(X_n = 0).$$

But we know that $\pi_0 := \lim_{n\to\infty} P(X_n = 0)$, thus

$$\pi \geq \lim_{n\to\infty} P(X_n = 0) = \pi_0.$$

and hence $\quad \pi_0 \leq \pi \quad$ for any solution π of $\pi = \sum_{j\geq 0} \pi^j P_j$.

But this means that π_0 is the smallest solution of $\pi = \sum_{j\geq 0} \pi^j P_j$ since π_0 also solves $\pi = \sum_{j\geq 0} \pi^j P_j$.

4.66

a) $\mu := \sum_{j\geq 0} j P_j = 0 \cdot \frac{1}{4} + 1 \cdot 0 + 2 \cdot \frac{3}{4} = \frac{3}{2}$

$\longrightarrow \quad \mu > 1 \quad \Rightarrow \quad \pi_0 < 1$

On the other hand, π_0 solves $\pi_0 = \sum_j (\pi_0)^j P_j$ and hence:

$$\pi_0 = \frac{1}{4} + \frac{3}{4}\pi_0^2 \quad \Rightarrow \quad \pi_0 = 1 \text{ or } \pi_0 = \frac{1}{3}$$

Thus $\pi_0 = \frac{1}{3}$

b) $\mu = 1 \quad \Rightarrow \quad \pi_0 = 1$

c) $\mu > 1 \quad \& \quad \pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^3$

$\Rightarrow \quad \pi_0 < 1 \quad \& \quad \left(\pi_0 = 1 \text{ or } \pi_0 = \frac{-1 \pm \sqrt{3}}{2}\right)$

$\Rightarrow \quad \pi_0 = \frac{-1+\sqrt{3}}{2}$.