Ex 5.35

$N(t)$ is a Poisson process with rate λ iff:

i) $N(0) = 0$

ii) $\{N(t)\}_{t\geq 0}$ has indep. increments

iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

$\forall\, t \geq 0$

Let's check that $\{N_s(t)\}_{t\geq 0}$ defined by $N_s(t) = N(s+t) - N(s)$ is a Poisson process.

i) $N_s(0) = N(0+s) - N(s) = 0$

ii) Take $t_1 \leq t_2 \leq t_3 \leq t_4$, then $\begin{cases} N_s(t_4) - N_s(t_3) = N(t_4+s) - N(t_3+s) \\ N_s(t_2) - N_s(t_1) = N(t_2+s) - N(t_1+s) \end{cases}$

but we have $t_1 + s \leq t_2 + s \leq t_3 + s \leq t_4 + s$ and hence

$N_s(t_4) - N_s(t_3)$ is indep. from $N_s(t_2) - N_s(t_1)$ by the independence of increment property for $\{N(t)\}_{t\geq 0}$.

iii) $P(N_s(t+h) - N_s(t) = 1) = P(N(s+t+h) - N(s+t) = 1)$

$= P(N(t'+h) - N(t') = 1) = \lambda h + o(h)$

iv) Similar to iii).

Ex 5.39 a) Let $N(t) :=$ # of mistakes in cell divisions in person at age t (in years).

Then $N(t) = \sum$ (of all mistakes in all divisions up to time t).

Let $T_0 := 0$ and $T_1 :=$ time of first mistake. Define

$T_n :=$ (time between $(n-1)$th mistake and nth mistake)

Then $S_n := \sum_{i=1}^{n} T_i$ is the arrival time of the n-th mistake.

Hence we should find $E[S_n]$.

In fact, $E[S_n] = E\left[\sum_{i=1}^{n} T_i\right] \overset{\text{idep}}{=} \sum_{i=1}^{n} E[T_i] = \dfrac{n}{\lambda} \quad (T_i \sim \exp(\lambda))$

Thus $E[S_{196}] = \dfrac{196}{2.5}$.

b) Similarly $Var[S_{196}] = \dfrac{196}{(2.5)^2}$

c) Noting that $S_n \sim \Gamma(n, \lambda) = \text{Gamma}(n, \lambda)$ (See page 302 in Ross's)

we get that $P(\text{an individual dies before } 67.2) = P(S_{196} \leq 67.2) = F_{S_{196}}(67.2)$

where $F_{S_{196}}$ is the CDF of S_{196} (the CDF of a Gamma$(196, \lambda)$)

d) $1 - F_{S_{196}}(90)$

e) $1 - F_{S_{196}}(100)$

Ex 5.40 $\{N_i(t)\}_{t\geq 0}$ is a Poisson process with rate λ_i. $(i=1,2)$

$\{N_1(t)\}$ is indep. of $\{N_2(t)\}$. Is $\{N_1(t)+N_2(t)\}_{t\geq 0}$ Poisson process with rate $\lambda_1+\lambda_2$?

i) $N_1(0)+N_2(0) = 0+0 = 0$

ii) From independence of $\{N_1(t)\}, \{N_2(t)\}$ combined with the independence of increments of each of the processes.

iii) $P\left((N_1+N_2)(t+h) - (N_1+N_2)(t) = 1\right)$

$= P\left[\left(N_1(t+h) - N_1(t)\right) + \left(N_2(t+h) - N_2(t)\right) = 1\right]$

$= P\left[\left(N_1(t+h) - N_1(t)\right) + \left(N_2(t+h) - N_2(t)\right) = 1 \mid N_2(t+h) - N_2(t) = 0\right] \times P\left(N_2(t+h) - N_2(t) = 0\right)$

$+ P\left[\left(N_1(t+h) - N_1(t)\right) + \left(N_2(t+h) - N_2(t)\right) = 1 \mid N_2(t+h) - N_2(t) = 1\right] \times P\left(N_2(t+h) - N_2(t) = 1\right)$

$= P\left[N_1(t+h) - N_1(t) = 1\right] \times P\left(N_2(t+h) - N_2(t) = 0\right) +$

$P\left[N_1(t+h) - N_1(t) = 0\right] \times P\left(N_2(t+h) - N_2(t) = 1\right)$

$= (\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (1 - \lambda_1 h + o(h)) \times (\lambda_2 h + o(h))$

$= (\lambda_1 + \lambda_2) h + o(h)$ (by the properties of $o(h)$).

iv) Similar to iii).

Ex 5.41 $P\left(N_1(t) = 1 \mid N_1(t) + N_2(t) = 1\right) = P\left(\min(T_1^1, T_1^2) = T_1^1\right)$

where T_1^1 is the time of the occurrence of the 1st event for $\{N_1(t)\}_{t\geq 0}$

T_1^2 " " " " $\{N_2(t)\}_{t\geq 0}$

but this is equal to $\frac{\lambda_1}{\lambda_1+\lambda_2}$ by the properties of exponentially distributed random variables.

Ex 5.42 a) $S_n = \sum_{i=1}^{n} T_i \implies E[S_n] = \sum_{i=1}^{n} E[T_i] = \frac{n}{\lambda}$

b) $E[S_4 \mid N(1) = 2] = E[$ 4 events will occur $\mid$ at time 1, 2 events have occurred$]$

(by memorylessness) $= 1 + E[$ 2 events will occur$] = 1 + E[S_2] = 1 + \frac{2}{\lambda}$.

c) $E[N(4) - N(2) \mid N(1) = 3] = E[N(4) - N(2) \mid N(1) - N(0) = 3]$

(indep. increments) $= E[N(4) - N(2)] = 2\lambda$.

Ex 5.45

* $$\mathrm{Cov}(T, N(T)) = E\big[(N(T) - E[N(T)])(T - E[T])\big]$$

$$= E[N(T)\cdot T] - E[T]\cdot E[N(T)] - E[T]\cdot E[N(T)] + E[N(T)]\cdot E[T]$$

$$= E\big[E[N(T)\cdot T \mid T]\big] - E[T]\cdot E\big[E[N(T) \mid T]\big]$$

$$= E\big[T\, E[N(T) \mid T]\big] - E[T]\cdot E[\lambda T].$$

$$= E[T\cdot \lambda T] - \lambda E[T]^2$$

$$= \lambda\left(E[T^2] - E[T]^2\right) = \lambda \sigma^2.$$

* $$\mathrm{Var}(N(T)) = E[\mathrm{Var}(N(T) \mid T)] + \mathrm{Var}(E[N(T) \mid T])$$

$$= E[\lambda T] + \mathrm{Var}(\lambda T) = \lambda \mu + \lambda^2 \sigma^2.$$