Ex 5.46 $\quad Cov\left(N(t), \sum_{i=1}^{N(t)} X_i\right) = E\left[N(t) \sum_{i=1}^{N(t)} X_i\right] - E[N(t)] \cdot E\left[\sum_{i=1}^{N(t)} X_i\right]$

$$= E\left[E\left[N(t) \sum_{i=1}^{N(t)} X_i \mid N(t)\right]\right] - t\lambda \cdot E\left[E\left[\sum_{i=1}^{N(t)} X_i \mid N(t)\right]\right]$$

$$= E[N(t) \cdot N(t) \cdot E[X_i]] - t\lambda \cdot E[N(t) \cdot E[X_i]]$$

$$= \mu \cdot (\lambda t + (\lambda t)^2) - t\lambda \cdot \mu \cdot t\lambda$$

$$= \mu \lambda t.$$

Ex 5.50 $\quad X$ is given in fact by $\quad X = N(T) \quad$ with $T \sim unif([0,1])$

$\mu := E[T] = \frac{1}{2}$

$\sigma := Var[T] = \frac{1}{12}$

Thus: $\quad E[X] = \lambda\mu = \frac{7}{2}$

$$Var[X] = \lambda\mu + \lambda^2\sigma^2 = \frac{7}{2} + \frac{49}{12} = \frac{91}{12}.$$

Ex 5.57 $\quad$ a) $P(N(t+1) - N(t) = 0) = \frac{\lambda^0}{0!} e^{-\lambda \cdot 1} = e^{-\lambda} = e^{-2}$.

b) $E[T_1 + T_2 + T_3 + T_4] + 12\text{ pm} = \frac{4}{\lambda} + 12\text{ pm} = 2 + 12\text{ pm}$

$= 14\text{ pm}$.

c) $1 - P(N(t+2) - N(t) \leq 1) = 1 - \underbrace{e^{-2\lambda}}_{0 \text{ events}} - \underbrace{(2\lambda) e^{-2\lambda}}_{1 \text{ event}}$.

Ex 5.64 $\quad$ a) By thm 5.2 in Ross, given $N(t) = n$, the events are uniformly distributed in the interval $[0, t]$. Let $U_i \sim Uniform[0,t]$ be the arrival time of i-th person out of the n persons. Then:

$$E[X \mid N(t) = n] = E\left[\sum_{i=1}^{n} (t - U_i)\right] \overset{(indep)}{=} n\, E[t - U_i] = N(t) \cdot t - \frac{t}{2} = N(t) \cdot \frac{t}{2}$$

b) Similar to a), $\quad Var[X \mid N(t)] = N(t)\, Var(t - U_i) = N(t) \cdot \frac{t^2}{12}$

c) $\quad Var(X) = E[Var[X \mid N(t)]] + Var[E[X \mid N(t)]] = \frac{t^2}{12} E[N(t)] + \frac{t^2}{2^2} \cdot Var[N(t)]$

$$= \frac{\lambda t^3}{3}.$$