Exam 2012, Problem 3:

c)

$$E[D] = \int_0^\infty x \, f_D(x) \, dx$$ where f_D is the density function of D.

Thus $$f_D(x) = \frac{d}{dx} F_D(x) = 2\pi\lambda x \, e^{-\pi\lambda x^2} \quad \text{for } x \geq 0$$

and $$E[D] = 2 \int_0^\infty \pi\lambda x^2 \, e^{-\pi\lambda x^2} \, dx$$

Put $\frac{u^2}{2} = \pi\lambda x^2$ so as to get to something that looks like a Gaussian

then $du = \sqrt{2}\sqrt{\pi\lambda} \, dx$.

Hence $$E[D] = \frac{\cancel{2}}{\sqrt{2}\sqrt{\pi\lambda}} \int_0^\infty \frac{u^2}{\cancel{2}} e^{-\frac{u^2}{2}} \, du$$

It suffices to note that $\left(u \longrightarrow u^2 e^{-\frac{u^2}{2}}\right)$ is symmetric arround 0.

and thus: $$\int_0^\infty u^2 e^{-u^2} \, du = \frac{1}{2} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} \, du = \frac{1}{2}\sqrt{2\pi} \, \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u^2 e^{-\frac{u^2}{2}} \, du$$

But $$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u^2 e^{-\frac{u^2}{2}} \, du = E[X^2] = \underbrace{Var(X)}_{=1} + E[X]^2$$

where $X \sim N(0,1)$ i.e standard Normal.

In conclusion:

$$E[D] = \frac{1}{\sqrt{2}\sqrt{\pi\lambda}} \cdot \frac{1}{2}\sqrt{2\pi} = \frac{1}{2\sqrt{\lambda}} \quad \square$$