STK2130 – Chapter 10.2

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10.2 Hitting Times, Max Variable and Ruin

Let $\{X(t) : t \geq 0\}$ be a Brownian motion process with variance parameter σ^2, and let:

$$T_a = \inf\{t > 0 : X(t) = a\} = \text{The first time the process hits } a.$$

We want to compute $P(T_a \leq t)$, where $a > 0$.

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We want to compute $P(T_a \leq t)$, where $a > 0$. In order to so, we instead consider $P(X(t) \geq a)$, and condition on the event $\{T_a \leq t\}$:

$$\begin{aligned} P(X(t) \geq a) = P(X(t) \geq a | T_a \leq t) P(T_a \leq t) \\ + P(X(t) \geq a | T_a > t) P(T_a > t) \end{aligned}$$

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Moreover, we obviously have:

$$P(X(t) \geq a | T_a > t) = 0$$

Hitting Times, Max Variable and Ruin (cont.)

Hence, we have:

$$P(X(t) \geq a) = \frac{1}{2} P(T_a \leq t)$$

and thus:

$$P(T_a \leq t) = 2 \cdot P(X(t) \geq a) = 2 \cdot P(\frac{X(t)}{\sigma\sqrt{t}} \geq \frac{a}{\sigma\sqrt{t}}) = 2 \cdot \Phi(-\frac{a}{\sigma\sqrt{t}})$$

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If $a < 0$, we can use a similar argument, and obtain:

$$P(T_a \leq t) = 2 \cdot P(X(t) \leq a) = 2 \cdot P(\frac{X(t)}{\sigma\sqrt{t}} \leq \frac{a}{\sigma\sqrt{t}}) = 2 \cdot \Phi(\frac{a}{\sigma\sqrt{t}})$$

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The formulas can be combined to:

$$P(T_a \leq t) = 2 \cdot \Phi(-\frac{|a|}{\sigma\sqrt{t}})$$

Hitting Times, Max Variable and Ruin (cont.)

NOTE:

$$\text{If } a > 0: \qquad \max_{0 \le s \le t} X(s) \ge a \quad \Leftrightarrow \quad T_a \le t$$

$$\text{If } a < 0: \qquad \min_{0 \le s \le t} X(s) \le a \quad \Leftrightarrow \quad T_a \le t$$

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Hence, if $a > 0$, we have:

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Similarly, if $a < 0$, we have:

$$P(\min_{0 \le s \le t} X(s) \le a) = P(T_a \le t) = 2 \cdot \Phi(\frac{a}{\sigma\sqrt{t}})$$

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Finally, we let $b < 0 < a$, and let $T = \min\{T_a, T_b\}$ where:

$$T_a = \inf\{t > 0 : X(t) = a\} = \text{The first time the process hits } a$$

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We want to calculate $P(T_a < T_b) = P(X(T) = a)$.

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We want to calculate $P(T_a < T_b) = P(X(T) = a)$.

Let $P(X(T) = a) = p$, and $P(X(T) = b) = 1 - P(X(T) = a) = 1 - p$.

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Since $E[X(t)] = 0$ for all $t \geq 0$, it follows that we in particular must have:

$$0 = E[X(T)] = a \cdot p + b \cdot (1 - p) = (a - b)p + b$$

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$$0 = E[X(T)] = a \cdot p + b \cdot (1 - p) = (a - b)p + b$$

By solving this equation with respect to p, we get:

$$P(T_a < T_b) = P(X(T) = a) = p = \frac{-b}{a - b} = \frac{|b|}{a + |b|}$$