STK2130 – Chapter 10.3

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10.3 Variations on Brownian Motion

- Brownian Motion with drift
- Geometric Brownian Motion

10.3.1 Brownian Motion with drift

We say that $\{X(t) : t \geq 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2 if:

- $X(0) = 0$
- $\{X(t) : t \geq 0\}$ has stationary and independent increments
- $X(t) \sim N(\mu t, \sigma^2 t), \quad t \geq 0.$

An equivalent definition is to let $\{B(t) : t \geq 0\}$ be a standard Brownian motion process, and then define:

$$X(t) = \sigma B(t) + \mu t$$

10.3.2 Geometric Brownian Motion

If $\{Y(t) : t \geq 0\}$ is a Brownian motion process drift coefficient μ and variance parameter σ^2, then the process $\{X(t) : t \geq 0\}$ defined by:

$$X(t) = e^{Y(t)}$$

is called a geometric Brownian motion process.

If $\{X(t) : t \geq 0\}$ is geometric Brownian motion process, we may compute:

$$\begin{aligned} E[X(t)|X(u), 0 \leq u \leq s] &= E[e^{Y(t)}|Y(u), 0 \leq u \leq s] \\ &= E[e^{Y(s)+Y(t)-Y(s)}|Y(u), 0 \leq u \leq s] \\ &= X(s)E[e^{Y(t)-Y(s)}|Y(u), 0 \leq u \leq s] \\ &= X(s)E[e^{Y(t)-Y(s)}] \end{aligned}$$

Geometric Brownian Motion (cont.)

If $W \sim N(\xi, \tau^2)$, the moment generating function of W is given by:

$$M_W(u) = E[e^{uW}] = e^{\xi u + \tau^2 u^2/2}$$

Since $(Y(t) - Y(s)) \sim N(\mu(t-s), \sigma^2(t-s))$, it follows that:

$$E[e^{Y(t)-Y(s)}] = M_{Y(t)-Y(s)}(1) = e^{\mu(t-s)+\sigma^2(t-s)/2}$$

Hence, we obtain:

$$E[X(t)|X(u), 0 \leq u \leq s] = X(s)e^{\mu(t-s)+\sigma^2(t-s)/2}$$

Geometric Brownian Motion (cont.)

Geometric Brownian motion is useful in the modeling of stock prices over time when the percentage changes are independent and identically distributed.

We let:

$$X_n = \text{the price of some stock at time } n, \quad n = 0, 1, 2, \ldots$$

$$Y_n = \frac{X_n}{X_{n-1}}, \quad n = 1, 2, \ldots$$

Then, it might be reasonable to assume that $Y_1, Y_2, \ldots$, are independent and identically distributed.

Then it follows that:

$$X_n = X_0 Y_1 Y_2 \cdots Y_n$$

Or equivalently, since stock prices typically are positive numbers:

$$\log(X_n) = \log(X_0) + \sum_{i=1}^{n} log(Y_i)$$

Geometric Brownian Motion (cont.)

By the central limit theorem $\log(X_n)$, being a sum of independent identically distributed variables, will be approximately normally distributed.

Hence, X_n can be approximated by a geometric Brownian motion.