STK2130 – Chapter 5.4.1

A. B. Huseby

Department of Mathematics
University of Oslo, Norway

The non-homogeneous Poisson Process

Definition

A counting process $\{N(t) : t \geq 0\}$ is said to be a non-homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$, if:

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ *has independent increments.*

(iii) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

We also introduce the mean value function $m(t)$ defined by:

$$m(t) = \int_0^t \lambda(u)du$$

The non-homogeneous Poisson Process (cont.)

Lemma (5.3)

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, then:

$$P(N(t) = 0) = e^{-m(t)}, \quad t \geq 0.$$

Corollary

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, and let T_1 be the time of the first event. Then we have:

$$P(T_1 > t) = P(N(t) = 0) = e^{-m(t)}, \quad t \geq 0.$$

Moreover, the density of T_1 is given by:

$$f_{T_1}(t) = \lambda(t)e^{-m(t)}, \quad t \geq 0.$$

The non-homogeneous Poisson Process (cont.)

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process, and $s > 0$, we define:

$$N_s(t) = N(s + t) - N(s).$$

Lemma (5.4)

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, then $\{N_s(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda_s(t) = \lambda(s + t)$, $t \geq 0$.

NOTE:

$$N_s(t - s) = N(t - s + s) - N(s) = N(t) - N(s)$$

The non-homogeneous Poisson Process (cont.)

The mean value function of $\{N_s(t) : t \geq 0\}$ is given by:

$$
\begin{aligned}
m_s(t) &= \int_0^t \lambda_s(u)du \\
&= \int_0^t \lambda(s+u)du \qquad \text{Subst.: } v = s+u,\ dv = du. \\
&= \int_s^{s+t} \lambda(v)dv \\
&= m(s+t) - m(s)
\end{aligned}
$$

NOTE:

$$m_s(t-s) = m(t-s+s) - m(s) = m(t) - m(s) = \int_s^t \lambda(u)du$$

The non-homogeneous Poisson Process (cont.)

Theorem (5.3)

If $\{N(t) : t \geq 0\}$ *is a non-homogeneous Poisson process with intensity function* $\lambda(t)$*, then:*

$$P(N(t) = n) = \frac{(m(t))^n}{n!} e^{-m(t)}, \quad t \geq 0, \quad n = 0, 1, 2, \ldots$$

PROOF: Induction with respect to n. By Lemma 5.3 the theorem holds for $n = 0$.

We then assume that we have shown that:

$$P(N(t) = n) = \frac{(m(t))^n}{n!} e^{-m(t)}, \quad t \geq 0,$$

and consider the probability $P(N(t) = n + 1)$.

The non-homogeneous Poisson Process (cont.)

In order to calculate this probability we condition on T_1, noting that if $s > t$, then obviously $P(N(t) = n + 1 | T_1 = s) = 0$.

$$\begin{aligned}
P(N(t) = n+1) &= \int_0^t P(N(t) = n+1 | T_1 = s)\, f_{T_1}(s) ds \\
&= \int_0^t P(N(t) = n+1 | T_1 = s)\, \lambda(s) e^{-m(s)} ds \\
&= \int_0^t P(N(t) - N(s) = n | T_1 = s)\, \lambda(s) e^{-m(s)} ds \\
&= \int_0^t P(N(t) - N(s) = n)\, \lambda(s) e^{-m(s)} ds \qquad \text{(Indep. incr.)} \\
&= \int_0^t P(N_s(t-s) = n)\, \lambda(s) e^{-m(s)} ds
\end{aligned}$$

The non-homogeneous Poisson Process (cont.)

By Lemma 5.4 and the induction hypothesis it follows that:

$$P(N_s(t-s)=n) = \frac{(m_s(t-s))^n}{n!}e^{-m_s(t-s)}$$
$$= \frac{(m(t)-m(s)))^n}{n!}e^{-(m(t)-m(s))}$$

By inserting this into the integral we get:

$$P(N(t)=n+1) = \int_0^t P(N_s(t-s)=n)\,\lambda(s)e^{-m(s)}ds$$
$$= \int_0^t \frac{(m(t)-m(s)))^n}{n!}e^{-(m(t)-m(s))}\,\lambda(s)e^{-m(s)}ds$$

The non-homogeneous Poisson Process (cont.)

Simplifying the integrand yields:

$$P(N(t) = n+1) = \int_0^t \frac{(m(t) - m(s)))^n}{n!} e^{-(m(t)-m(s))} \lambda(s) e^{-m(s)} ds$$

$$= \frac{e^{-m(t)}}{n!} \int_0^t (m(t) - m(s))^n \lambda(s) ds$$

Substitute: $u = m(t) - m(s) = \int_s^t \lambda(v)dv$ and $du = -\lambda(s)ds$, and get:

$$P(N(t) = n+1) = -\frac{e^{-m(t)}}{n!} \int_{m(t)}^0 u^n du = \frac{e^{-m(t)}}{n!} \int_0^{m(t)} u^n du$$

$$= \frac{e^{-m(t)}}{n!} \cdot \frac{(m(t))^{n+1}}{n+1} = \frac{(m(t))^{n+1}}{(n+1)!} e^{-m(t)}$$

which completes the induction proof ■

The non-homogeneous Poisson Process (cont.)

REMARK

We recall that $\{N_s(t) : t \geq 0\}$ is a non-homogeneous Poisson Process with mean function:

$$m_s(t) = m(s+t) - m(s) = \int_0^{s+t} \lambda(u)du - \int_0^{s} \lambda(u)du = \int_s^{s+t} \lambda(u)du$$

By Theorem 5.3 this implies that $N_s(t) = N(s+t) - N(s) \sim Po(m_s(t))$, and:

$$E[N_s(t)] = m_s(t) = \int_s^{s+t} \lambda(u)du.$$

Moreover, $N_s(t-s) = N(t) - N(s) \sim Po(m_s(t-s)) = Po(m(t) - m(s))$, and:

$$E[N_s(t-s)] = m_s(t-s) = m(t) - m(s) = \int_s^{t} \lambda(u)du$$

Example 5.24

Hot dog stand, opens at 08 A.M., closes at 05 P.M.

- 08 A.M. — 11 A.M.: Steadily increasing intensity from 5 to 20
- 11 A.M. — 01 P.M.: Constant intensity of 20
- 01 P.M. — 05 P.M.: Steadily decreasing intensity from 20 to 12

By letting $t = 0$ represent 8 a.m. the customer arrival intensity function, denoted $\lambda(t)$, can be expressed as:

$$\lambda(t) = \begin{cases} 5 + 5t & 0 \leq t \leq 3 \\ 20 & 3 \leq t \leq 5 \\ 20 - 2(t - 5) & 5 \leq t \leq 9 \end{cases}$$

Hence, we have:

$$\lambda(0) = 5 + 5 \cdot 0 = 5, \qquad \lambda(3) = 5 + 5 \cdot 3 = 20,$$

$$\lambda(5) = 20 - 2 \cdot (5 - 5) = 20, \qquad \lambda(9) = 20 - 2 \cdot (9 - 5) = 12.$$

Example 5.24 (cont.)

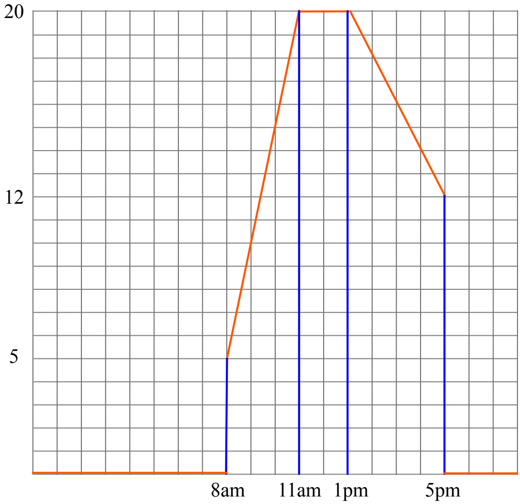

Figure: Customer arrival intensity function

Example 5.24 (cont.)

For $0 \leq s \leq t$ we let:

$$N(t) = \text{Number of arrivals at the hot dog stand in } [0, t]$$

$$N(s,t) = \text{Number of arrivals at the hot dog stand in } [s, t] = N(t) - N(s).$$

By Theorem 5.3 $N(t) \sim Po(m(t))$ and $N(s,t) \sim Po(m(t) - m(s))$, where:

$$m(t) = \int_0^t \lambda(u)du, \quad 0 \leq t$$

$$m(t) - m(s) = \int_s^t \lambda(u)du, \quad 0 \leq s \leq t$$

Example 5.24 (cont.)

Ex. A. What is the expected number of arrivals between 8:30 A.M. and 9:30 A.M? What is the probability that no customers arrive in this period?

SOLUTION: 8:30 A.M. and 9:30 A.M. correspond to respectively $s = \frac{1}{2}$ and $t = \frac{3}{2}$. From this we get that:

$$E[N(\tfrac{1}{2}, \tfrac{3}{2})] = m(\tfrac{3}{2}) - m(\tfrac{1}{2}) = \int_{1/2}^{3/2} \lambda(u)du = \int_{1/2}^{3/2} (5 + 5u)du$$

$$= \Big|_{(1/2)}^{(3/2)} (5t + \tfrac{5}{2}t^2)$$

$$= (\tfrac{15}{2} + \tfrac{45}{8}) - (\tfrac{5}{2} + \tfrac{5}{8}) = \tfrac{105}{8} - \tfrac{25}{8} = 10.$$

Moreover, we have:

$$P(N(\tfrac{1}{2}, \tfrac{3}{2}) = 0) = e^{-(m(t)-m(s))} = e^{-10} \approx 0.000045$$

Example 5.24 (cont.)

Ex. B. What is the expected number of arrivals between 1:00 P.M. and 3:00 P.M? What is the probability that at least two customers arrive in this period?

SOLUTION: 1:00 P.M. and 3:00 P.M correspond to respectively $s = 5$ and $t = 7$. From this we get that:

$$E[N(5,7)] = m(7) - m(5) = \int_5^7 \lambda(u)du = \int_5^7 (20 - 2(u-5))du$$

$$= \int_5^7 (20 - 2u + 10)du = |_5^7 (20t - t^2 + 10t))$$

$$= (140 - 49 + 70) - (100 - 25 + 50)$$

$$= 161 - 125 = 36.$$

Example 5.24 (cont.)

Moreover, we have:

$$\begin{aligned} P(N(5,7) \geq 2) &= 1 - P(N(5,7) \leq 1) \\ &= 1 - [P(N(5,7) = 0) + P(N(5,7) = 1)] \\ &= 1 - [e^{-(m(7)-m(5))} + \frac{(m(7) - m(5))^1}{1!} e^{-(m(7)-m(5))}] \\ &= 1 - [1 + \frac{36}{1}] \cdot e^{-36} \\ &= 1 - 37 \cdot e^{-36} \approx 1 \end{aligned}$$

Time sampling a homogeneous Poisson Process

We recall the following result (slightly modified):

Theorem (5.2)

We consider a Poisson process $\{N(t) : t \geq 0\}$, and assume that $N(t) - N(s) = n$, where $s < t$. Then the arrival times $S_1 < S_2 < \cdots < S_n$ in $(s, t]$ has the following joint density:

$$f(s_1, s_2, \ldots, s_n | N(t) - N(s) = n) = \frac{n!}{(t-s)^n}, \qquad s < s_1 < s_2 < \cdots < s_n < t.$$

The preceding result is often stated as follows:

Corollary (5.2)

Given that n events have occurred in the interval $(s, t]$, the times at which the events occur, considered as unordered random variables, are distributed independently and uniformly in the interval $(s, t]$.

Time sampling a Poisson Process (cont.)

Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson Process with rate λ where each event can be classified as either a Type 1 event or a Type 2 event.

If an event occurs at time t, then the probability that it is of type 1 is $p_1(t)$, and the probability that it is of type 2 is $p_2(t) = 1 - p_1(t)$.

We assume that the event type at time t is independent of the history of the Poisson process up to time t, and introduce:

$$N_i(t) = \text{The number of events of type } i \text{ in } [0, t] \quad t \geq 0, \quad i = 1, 2.$$

NOTE: $N(t) = N_1(t) + N_2(t)$. Moreover, it can be shown that for $s < t$:

$$(N_i(t) - N_i(s) | N(t) - N(s) = n) \sim Bin(n, \bar{p}_i(s, t)), \quad i = 1, 2,$$

where:

$$\bar{p}_i(s, t) = \frac{1}{t - s} \int_s^t p_i(u) du, \quad i = 1, 2.$$

Time sampling a Poisson Process (cont.)

In order to explain this, we recall that conditional on the event that $N(t) - N(s) = n$, the n arrival times in the interval $(s, t]$ are independent and uniformly distributed.

Given that the arrival time of an event is u, the probability that this event is of type i is $p_i(u)$. Hence, the unconditional probability that the event is of type i is:

$$P(\text{Type } i \text{ event}) = \int_s^t p_i(u) \frac{1}{t-s} du = \bar{p}_i(s,t).$$

Since all event types are classified independent of the Poisson process, we have a series of n binomial experiments with the same probability of success.

Hence, we have:

$$(N_i(t) - N_i(s) | N(t) - N(s) = n) \sim Bin(n, \bar{p}_i(s,t)), \quad i = 1, 2 \qquad \blacksquare$$

Time sampling a Poisson Process (cont.)

We will now show that:

- $\{N_i(t) : t \geq 0\}$ is a non-homogeneous Poisson Process with intensity function $\lambda p_i(t)$, $i = 1, 2$.
- $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent of each other.

In order to show the first claim, we verify that $\{N_i(t) : t \geq 0\}$ satisfies the axioms.

The proof of the second claim is similar to the corresponding result for the case where the probabilities $p_1(t)$ and $p_2(t)$ are constant.

Time sampling a Poisson Process (cont.)

PROOF: Since $N(0) = 0$, it follows that $N_i(0) = 0$, $i = 1, 2$ as well.

Let $(s_1, t_1]$ and $(s_2, t_2]$ be disjoint. Since $\{N(t) : t \geq 0\}$ has independent increments, we have for $i = 1, 2$:

$$
\begin{aligned}
&P(N_i(t_2) - N_i(s_2) = k | N_i(t_1) - N_i(s_1) = \ell) \\
&\quad = \sum_{n=k}^{\infty} P(N_i(t_2) - N_i(s_2) = k | N(t_2) - N(s_2) = n, N_i(t_1) - N_i(s_1) = \ell) \\
&\quad \cdot P(N(t_2) - N(s_2) = n | N_i(t_1) - N_i(s_1) = \ell) \\
&\quad = \sum_{n=k}^{\infty} P(N_i(t_2) - N_i(s_2) = k | N(t_2) - N(s_2) = n) \cdot P(N(t_2) - N(s_2) = n) \\
&\quad = P(N_i(t_2) - N_i(s_2) = k)
\end{aligned}
$$

Hence, $\{N_i(t) : t \geq 0\}$ have independent increments, $i = 1, 2$.

Time sampling a Poisson Process (cont.)

Moreover, we have:

$$
\begin{aligned}
&P(N_1(t+h) - N_1(t) = 1) \\
&\quad = P(N_1(t+h) - N_1(t) = 1 | N(t+h) - N(t) = 1) \cdot P(N(t+h) - N(t) = 1) \\
&\quad + P(N_1(t+h) - N_1(t) = 1 | N(t+h) - N(t) \geq 2) \cdot P(N(t+h) - N(t) \geq 2) \\
&\quad = p_1(t) \cdot (\lambda h + o(h)) + o(h) \\
&\quad = \lambda p_1(t) h + o(h)
\end{aligned}
$$

and:

$$P(N_1(t+h) - N_1(t) \geq 2) \leq P(N(t+h) - N(t) \geq 2) = o(h).$$

By similar arguments we get that:

$$
\begin{aligned}
P(N_2(t+h) - N_2(t) = 1) &= \lambda p_2(t) h + o(h) \\
P(N_2(t+h) - N_2(t) \geq 2) &= o(h).
\end{aligned}
$$

Time sampling a Poisson Process (cont.)

Before we prove that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent, we note that, as a consequence of the first part of the proof, we have for $s < t$ that:

$$P(N_i(t) - N_i(s) = k) = \frac{(m_i(t) - m_i(s))^k}{k!} e^{-(m_i(t) - m_i(s))}, \quad k = 0, 1, 2, \ldots$$

where:

$$\begin{aligned} m_i(t) - m_i(s) &= \int_0^t \lambda p_i(u) du - \int_0^s \lambda p_i(u) du \\ &= \int_s^t \lambda p_i(u) du = \lambda(t - s) \int_s^t p_i(u) \frac{1}{t - s} du = \lambda \bar{p}_i(s, t)(t - s). \end{aligned}$$

Note also that since $p_1(t) + p_2(t) = 1$, we also have $\bar{p}_1(s, t) + \bar{p}_2(s, t) = 1$.

Time sampling a Poisson Process (cont.)

To show that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent we let $s < t$, and consider:

$$P[(N_1(t) - N_1(s) = k) \ \cap \ (N_2(t) - N_2(s) = \ell)]$$

$$= P[(N_1(t) - N_1(s) = k) \ \cap \ (N(t) - N(s) = k + \ell)]$$

$$= \binom{k+\ell}{k} \bar{p}_1(s,t)^k \cdot \bar{p}_2(s,t)^\ell \ \cdot \ \frac{[\lambda(t-s)]^{k+\ell}}{(k+\ell)!} e^{-\lambda(t-s)}$$

$$= \frac{(\lambda \bar{p}_1(s,t)(t-s))^k}{k!} e^{-\lambda \bar{p}_1(s,t)(t-s)} \ \cdot \ \frac{(\lambda \bar{p}_2(s,t)(t-s))^\ell}{\ell!} e^{-\lambda \bar{p}_2(s,t)(t-s)}$$

$$= P(N_1(t) - N_1(s) = k) \ \cdot \ P(N_2(t) - N_2(s) = \ell)$$

Hence, we conclude that $(N_1(t) - N_1(s))$ and $(N_2(t) - N_2(s))$ are independent for all $s < t$, implying that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent.

Example 5.25 – An $M/G/\infty$ queue

Clients arrive at a server according to homogeneous Poisson process $\{N(t) : t \geq 0\}$ with rate λ:

$$N(t) = \text{Number of clients arriving in } [0, t], \quad t \geq 0.$$

We then introduce:

$$X_n = \text{The amount time it takes to serve the } n\text{th client}, \quad n = 1, 2, \ldots$$

We assume that $X_1, X_2, \ldots$ are independent and identically distributed with cumulative distribution function G, and also that $X_1, X_2, \ldots$ are independent of $\{N(t) : t \geq 0\}$.

Example 5.25 (cont.)

An $M/G/c$ queue is a stochastic process where:

- Clients arrive according to a Markovian counting process, (i.e. a Poisson process), which explains the M in the notation
- The amount time it takes to serve a client has cdf G
- The server has a capacity of c, i.e., c clients can be served at the same time.

In this case $X_1, X_2, \ldots$ are independent and identically distributed, which is justified by assuming that the server has an infinite capacity, i.e., $c = \infty$. Thus, the time it takes to serve a client is not affected by the number of clients presently being served.

So in this particular case we have an $M/G/\infty$ queue.

Example 5.25 (cont.)

We then assume that $0 \leq s < t$, and introduce:

$$D(s,t) = \text{Number of clients departing in } (s,t].$$

In order to find the probability distribution of $D(s,t)$ we start by arguing that:

$$(D(s,t)|N(t) = n) \sim Bin(n, \bar{p}(s,t)), \quad t \geq 0,$$

where:

$$\bar{p}(s,t) = \frac{1}{t} \int_s^t G(u)du.$$

In order to explain this, we recall that conditional on the event that $N(t) = n$, the n arrival times in the interval $[0,t]$ are independent and uniformly distributed.

Example 5.25 (cont.)

Given that the arrival time of a client is $u \in [0, t]$ and denoting the service time by X, the probability that this client departs in the interval $(s, t]$ is:

$$P(s < u + X \leq t | u) = \begin{cases} G(t-u) - G(s-u) & \text{if } u < s \\ G(t-u) & \text{if } s \leq u < t \end{cases}$$

Hence, the unconditional probability that the client departs in $[s, t]$ is:

$$P(s < u + X \leq t) = \frac{1}{t} \cdot [\int_0^s (G(t-u) - G(s-u))du + \int_s^t G(t-u)du]$$

$$= \frac{1}{t} \cdot [\int_0^t G(t-u)du - \int_0^s G(s-u)du] \quad \text{(Subst.: } v = t-u \text{ and } v = s-u.)$$

$$= \frac{1}{t} \cdot [\int_0^t G(v)dv - \int_0^s G(v)dv] = \frac{1}{t} \int_s^t G(v)dv = \bar{p}(s, t)$$

Example 5.25 (cont.)

Since X is independent of the Poisson process, we have a series of n binomial experiments with the same probability of success.

Hence, we have $(D(s,t)|N(t) = n) \sim Bin(n, \bar{p}(s,t))$, ■

In the following we let $D(t) = D(0,t)$, and claim that $\{D(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function:

$$\lambda(t) = \lambda \cdot G(t).$$

In order to show this, we must verify that the axioms (i), (ii), (iii) and (iv) of the definition are satisfied. Axiom (i) states that $D(0) = 0$, which is obviously satisfied.

In order to verify the other axioms we first find the unconditional probability distribution of $D(s,t)$.

Example 5.25 (cont.)

The probability distribution of $D(s,t)$ is obtained by conditioning on $N(t)$:

$$
\begin{aligned}
P(D(s,t)=k) &= \sum_{n=k}^{\infty} P(D(s,t)=k|N(t)=n)\cdot P(N(t)=n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} [\bar{p}(s,t)]^k [1-\bar{p}(s,t)]^{n-k} \cdot \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \frac{(\bar{p}(s,t)\lambda t)^k}{k!} e^{-\bar{p}(s,t)\lambda t} \sum_{n=k}^{\infty} \frac{((1-\bar{p}(s,t))\lambda t)^{n-k}}{(n-k)!} e^{-(1-\bar{p}(s,t))\lambda t} \\
&= \frac{(\bar{p}(s,t)\lambda t)^k}{k!} e^{-\bar{p}(s,t)\lambda t} \sum_{j=0}^{\infty} \frac{((1-\bar{p}(s,t))\lambda t)^{j}}{j!} e^{-(1-\bar{p}(s,t))\lambda t} \\
&= \frac{(\bar{p}(s,t)\lambda t)^k}{k!} e^{-\bar{p}(s,t)\lambda t}, \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

Example 5.25 (cont.)

Thus, we conclude that $D(s,t) \sim Po(\bar{p}(s,t) \cdot \lambda t)$. By a similar argument it can also be shown that if $(s_1, t_1]$ and $(s_2, t_2]$ are disjoint intervals, then $D(s_1, t_1)$ and $D(s_2, t_2)$ are independent.

We recall that:

$$\bar{p}(s,t) = \frac{1}{t} \int_s^t G(u)du.$$

Hence, we get that:

$$\bar{p}(s,t) \cdot \lambda t = \left[\frac{1}{t} \int_s^t G(u)du \right] \cdot \lambda t = \int_s^t \lambda G(u)du.$$

From this we get that:

$$D(s,t) \sim Po(\int_s^t \lambda G(u)du)$$

Example 5.25 (cont.)

By Taylor expansion we have for a given function f and $h > 0$ that:

$$f(h) = f(0) + f'(0)h + o(h)$$

Hence, we get:

$$f_1(h) = \int_t^{t+h} \lambda G(u) du = \lambda G(t)h + o(h)$$

$$f_2(h) = e^{-ah} = 1 - ah + o(h)$$

Hence, we get:

$$P(D(t, t+h) = 0) = e^{-(\lambda G(t)h + o(h))} = 1 - \lambda G(t)h + o(h)$$

$$P(D(t, t+h) = 1) = \frac{\lambda G(t)h + o(h)}{1!} e^{-(\lambda G(t)h + o(h))} = \lambda G(t)h + o(h)$$

$$P(D(t, t+h) \geq 2) = 1 - [1 - \lambda G(t)h + o(h) + \lambda G(t)h + o(h)] = o(h)$$

Example 5.25 (cont.)

By combining all the above results, it follows that $\{D(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function:

$$\lambda(t) = \lambda \cdot G(t).$$

NOTE:

$$\lim_{t \to \infty} \lambda(t) \; = \; \lambda \cdot \lim_{t \to \infty} G(t) \; = \; \lambda.$$

Hence, when t is large, the intensity function of the departure process $\{D(t) : t \geq 0\}$ is approximately equal to the arrival rate λ.

The non-homogeneous Poisson process (cont.)

Let $\{N(t) : t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$, and mean value function $m(t)$. Furthermore, let:

$$S_n = \text{The time of the } n\text{th event}, \quad n = 1, 2, \ldots$$

We have shown that the density of $S_1 = T_1$ is given by:

$$f_{S_1}(t) = \lambda(t) e^{-m(t)}, \quad t \geq 0.$$

We shall now derive the density of S_n, $n = 1, 2, \ldots$.

In order to do so, it is convenient once again to introduce:

$$N(s, t) = N(t) - N(s), \quad 0 \leq s < t$$

The non-homogeneous Poisson process (cont.)

Let $h > 0$. We then have:

$$
\begin{aligned}
P(t < S_n \leq t+h) &= P(N(t) = n-1 \cap N(t, t+h) = 1) + o(h) \\
&= P(N(t) = n-1) \cdot P(N(t, t+h) = 1) + o(h) \\
&= \frac{[m(t)]^{n-1}}{(n-1)!} e^{-m(t)} \cdot [\lambda(t)h + o(h)] + o(h) \\
&= \frac{[m(t)]^{n-1}}{(n-1)!} \lambda(t) e^{-m(t)} h + o(h)
\end{aligned}
$$

Hence, the density of S_n becomes:

$$
f_{S_n}(t) = \lim_{h \to 0} \frac{P(t < S_n \leq t+h)}{h} = \lim_{h \to 0} \left[\frac{[m(t)]^{n-1}}{(n-1)!} \lambda(t) e^{-m(t)} + \frac{o(h)}{h} \right]
$$

$$
= \frac{[m(t)]^{n-1}}{(n-1)!} \lambda(t) e^{-m(t)}.
$$

The non-homogeneous Poisson process (cont.)

NOTE 1. If $n = 1$, we as before get:

$$f_{S_1}(t) = \frac{[m(t)]^{1-1}}{(1-1)!}\lambda(t)e^{-m(t)} = \lambda(t)e^{-m(t)}.$$

NOTE 2. If $\lambda(t) = \lambda$, then $m(t) = \int_0^t \lambda du = \lambda t$, and we get:

$$f_{S_n}(t) = \frac{[m(t)]^{n-1}}{(n-1)!}\lambda(t)e^{-m(t)}$$

$$= \frac{[\lambda t]^{n-1}}{(n-1)!}\lambda e^{-\lambda t} = \frac{\lambda^n}{\Gamma(n)}t^{n-1}e^{-\lambda t}$$

Thus, in this case $S_n \sim Gamma(n, \lambda)$ as before.