STK2130 – Chapter 5.4.2

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Mean and variance of a Poisson distributed variable

We start out by showing how the mean and variance of a Poisson variable can be calculated, and assume that $X \sim Po(\mu)$. Thus, the probability distribution of X is given by:

$$P(X = x) = \frac{\mu^x}{x!} e^{-\mu}, \quad x = 0, 1, 2, \ldots$$

In order to find the mean and the variance of X, we determine the moment generating function:

$$M_X(t) = E[e^{tX}] \;=\; \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\mu^x}{x!} e^{-\mu} \;=\; \sum_{x=0}^{\infty} \frac{(\mu \cdot e^t)^x}{x!} e^{-\mu}$$

$$= e^{-\mu} \cdot e^{\mu e^t} \cdot \sum_{x=0}^{\infty} \frac{(\mu \cdot e^t)^x}{x!} e^{-\mu e^t} \;=\; e^{\mu(e^t - 1)}.$$

Mean and variance (cont.)

The first derivative of the moment generating function is:

$$\frac{\partial}{\partial t} M_X(t) = \frac{\partial}{\partial t} E[e^{tX}] = E\left[\frac{\partial}{\partial t} e^{tX}\right] = E[e^{tX} \cdot X]$$

By inserting $t = 0$, we get:

$$\frac{\partial}{\partial t} M_X(0) = E[e^{0 \cdot X} \cdot X] = E[X].$$

Moreover, the second order derivative of $M_X(t)$ is:

$$\frac{\partial^2}{\partial t^2} M_X(t) = \frac{\partial^2}{\partial t^2} E[e^{tX}] = E\left[\frac{\partial^2}{\partial t^2} e^{tX}\right] = E[e^{tX} \cdot X^2]$$

By inserting $t = 0$, we get:

$$\frac{\partial^2}{\partial t^2} M_X(0) = E[e^{0 \cdot X} \cdot X^2] = E[X^2].$$

Mean and variance (cont.)

We then use this to calculate $E[X]$ and $E[X^2]$ in the case where $X \sim Po(\mu)$.

$$\frac{\partial}{\partial t} M_X(t) = \frac{\partial}{\partial t} e^{\mu(e^t - 1)} = e^{\mu(e^t - 1)} \cdot \mu e^t,$$

implying that:

$$E[X] = \frac{\partial}{\partial t} M_X(0) = e^{\mu(e^0 - 1)} \cdot \mu e^0 = \mu$$

Moreover,

$$\frac{\partial^2}{\partial t^2} M_X(t) = \frac{\partial^2}{\partial t^2} e^{\mu(e^t - 1)} = e^{\mu(e^t - 1)} \cdot (\mu e^t)^2 + e^{\mu(e^t - 1)} \cdot \mu e^t,$$

implying that:

$$E[X^2] = \frac{\partial^2}{\partial t^2} M_X(0) = e^{\mu(e^0 - 1)} \cdot (\mu e^0)^2 + e^{\mu(e^0 - 1)} \cdot \mu e^0 = \mu^2 + \mu.$$

Mean and variance (cont.)

Hence, $\mathrm{Var}[X]$ becomes:

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \mu^2 + \mu - \mu^2 = \mu.$$

Thus, if $X \sim Po(\mu)$, then $E[X] = \mathrm{Var}[X] = \mu$.

If $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λ, we have shown that:

$$N(t) \sim Po(\lambda t).$$

Hence, we get that:

$$E[N(t)] = \mathrm{Var}[N(t)] = \lambda t$$

Sums of Poisson variables

Let $X_1, \ldots, X_n$ be independent and assume that $X_i \sim Po(\mu_i)$, $i = 1, \ldots, n$. We then consider:

$$S = \sum_{i=1}^{n} X_i$$

Since the X_is are independent, the moment generating function of S is given by:

$$\begin{aligned} M_S(t) = E[e^{tS}] \; &= \; E[e^{tX_1 + \cdots + tX_n}] \; = \; E[e^{tX_1}] \cdots E[e^{tX_n}] \\ &= e^{\mu_1(e^t - 1)} \cdots e^{\mu_n(e^t - 1)} \; = \; e^{(\mu_1 + \cdots + \mu_n)(e^t - 1)} \end{aligned}$$

This is the moment generating function of a $Po(\mu_1 + \cdots + \mu_n)$-distribution. Thus, we conclude that:

$$S \sim Po(\mu_1 + \cdots + \mu_n).$$

Compound Poisson Process

Let $\{N(t) : t \geq 0\}$ be a Poisson process, and let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed variables, and independent of $\{N(t) : t \geq 0\}$.

We then define a new stochastic process $\{X(t) : t \geq 0\}$ such that:

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

The process $\{X(t) : t \geq 0\}$ is said to be a compound Poisson process.

NOTE: If $P(Y_i = 1) = 1$, $i = 1, 2, \ldots$, then obviously $X(t) = N(t)$. Thus, a (regular) Poisson process is a special case of a compound Poisson process.

Compound Poisson Process (cont.)

EXAMPLE 1. An insurance company receives claims from its clients at random points of time. We let:

$$N(t) = \text{The number of claims in } [0, t], \quad t \geq 0$$

and assume that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Moreover, we let:

$$Y_i = \text{The size in NOK of the } i\text{th claim}, \quad i = 1, 2, \ldots,$$

and assume that $Y_1, Y_2, \ldots$ are independent and identically distributed variables and independent of $\{N(t) : t \geq 0\}$.

We then introduce:

$$X(t) = \sum_{i=1}^{N(t)} Y_i = \text{The sum of claims in } [0, t], \quad t \geq 0.$$

Then $\{X(t) : t \geq 0\}$ is a compound Poisson process.

Compound Poisson Process (cont.)

EXAMPLE 2. A stock is traded at random points in time. We let:

$$N(t) = \text{The number of trades in } [0, t], \quad t \geq 0$$

and assume that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Moreover, we let:

$$Y_i = \text{The change in stock price the } i\text{th time the stock is traded}, \quad i = 1, 2, \ldots,$$

and assume that $Y_1, Y_2, \ldots$ are independent and identically distributed variables and independent of $\{N(t) : t \geq 0\}$.

We then introduce:

$$X(t) = \sum_{i=1}^{N(t)} Y_i = \text{The cumulative change in stock price } [0, t], \quad t \geq 0.$$

Then $\{X(t) : t \geq 0\}$ is a compound Poisson process.

Compound Poisson Process (cont.)

Let $E[Y_i] = \mu$ and $E[Y_i^2] = \nu$, $i = 1, 2, \ldots$. Thus, $\mathrm{Var}[Y_i] = \nu - \mu^2$.

The expectation and variance of $X(t)$, calculated by conditioning on $N(t)$ is:

$$E[X(t)] = E[E[\sum_{i=1}^{n} Y_i | N(t) = n]] \;=\; E[N(t)\mu] \;=\; \lambda t \cdot \mu \;=\; \lambda t \cdot E[Y_i]$$

$$\begin{aligned}
\mathrm{Var}[X(t)] &= \mathrm{Var}[E[\sum_{i=1}^{n} Y_i | N(t) = n]] + E[\mathrm{Var}[\sum_{i=1}^{n} Y_i | N(t) = n]] \\
&= \mathrm{Var}[N(t)\mu] + E[N(t)(\nu - \mu^2)] \;=\; \mu^2 \,\mathrm{Var}[N(t)] + (\nu - \mu^2) E[N(t)] \\
&= \mu^2 \lambda t + (\nu - \mu^2)\lambda t \;=\; \lambda t \cdot \nu \;=\; \lambda t \cdot E[Y_i^2]
\end{aligned}$$

Example 5.26

Families migrating to an area following a Poisson process, $\{N(t) : t \geq 0\}$, with rate $\lambda = 2$ per week.

$$N(t) = \text{Number of families migrating in } [0, t], \quad t \geq 0.$$

$$Y_i = \text{Number of people in the } i\text{th family}, \quad i = 1, 2, \ldots$$

We assume that the probability distribution for the Y_is is given by:

$$P(Y_i = 1) = \tfrac{1}{6}, \quad P(Y_i = 2) = \tfrac{2}{6}, \quad P(Y_i = 3) = \tfrac{2}{6}, \quad P(Y_i = 4) = \tfrac{1}{6}.$$

Hence, we get:

$$E[Y_i] = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{2}{6} + 3 \cdot \tfrac{2}{6} + 4 \cdot \tfrac{1}{6} = \frac{1}{6}[1 + 4 + 6 + 4] = \tfrac{15}{6} = \tfrac{5}{2}$$

$$E[Y_i^2] = 1^2 \cdot \tfrac{1}{6} + 2^2 \cdot \tfrac{2}{6} + 3^2 \cdot \tfrac{2}{6} + 4^2 \cdot \tfrac{1}{6} = \frac{1}{6}[1 + 8 + 18 + 16] = \tfrac{43}{6}$$

Example 5.26 (cont.)

We then consider the compound Poisson process $\{X(t) : t \geq 0\}$, where:

$$X(t) = \text{The number of people migrating in } [0, t], \quad f \geq 0.$$

We then have:

$$E[X(5)] \;=\; \lambda \cdot 5 \cdot E[Y_i] \;=\; 2 \cdot 5 \cdot \tfrac{15}{6} \;=\; \tfrac{150}{6} \;=\; 25$$

$$\mathrm{Var}[X(5)] \;=\; \lambda \cdot 5 \cdot E[Y_i^2] \;=\; 2 \cdot 5 \cdot \tfrac{43}{6} \;=\; \tfrac{430}{6} \;=\; \tfrac{215}{3}.$$

Example 5.27 – An $M/G/1$ queue

We consider a server with capacity $c = 1$. Thus, the server can serve one client at a time. The clients arrive according to a homogeneous Poisson process with rate λ.

We assume that the clients are served according to a first come, first served rule. That is, clients are served in the order in which they arrive, those who arrive first are served first.

We let:

$$S_i = \text{The time it takes to serve the } i\text{th client}, \qquad i = 1, 2, \ldots$$

Since we are considering an $M/G/1$ queue, $S_1, S_2, \ldots$ are independent and identically distributed variables with cumulative distribution G.

In this case, however, we will only consider the mean and standard deviation of this distribution:

$$E[S_i] = \mu, \qquad \mathrm{Var}[S_i] = \sigma^2.$$

Example 5.27 (cont.)

We let:

$$B_i = \text{The time from the service of the } i\text{th client starts,}$$
$$\text{until the queue is empty}, \quad i = 1, 2, \ldots$$

Since the clients arrive according to a homogeneous Poisson process, and since we apply a first come, first served rule, all the B_is are identically distributed unless the size of the queue explodes.

Our goal is to find $E[B]$ and $\mathrm{Var}[B]$, where B is a random variable having the same distribution as the B_is.

Assuming that the ith client arrives at time u, we then argue that:

$$B_i = S_i + \sum_{j=1}^{N_u(S_i)} B_{i+j},$$

where $N_u(t) = N(t) - N(u)$ as usual.

Example 5.27 (cont.)

NOTE 1. $N_u(S_i)$ is the number of clients arriving while the ith client is being served. Each of these clients starts a new busy period for the server, and all these periods have to be added to the time it takes to serve the ith client before the queue is empty.

NOTE 2. $\{N_u(t) : t \geq 0\}$ is also a homogeneous Poisson process with rate λ. Hence, the process $\{X_u(t) : t \geq 0\}$, where:

$$X_u(t) = \sum_{j=1}^{N_u(t)} B_{i+j}, \quad t \geq 0,$$

is a compound Poisson process.

Example 5.27 (cont.)

By conditioning on S_i we then get:

$$E[B_i|S_i] = S_i + E\left[\sum_{j=1}^{N_u(S_i)} B_{i+j} \,|\, S_i\right] = S_i + \lambda S_i E[B] = (1 + \lambda E[B])S_i$$

Hence, it follows that:

$$E[B] = E[E[B_i|S_i]] = (1 + \lambda E[B])E[S_i] = (1 + \lambda E[B])\mu$$

We then try to solve this equation with respect to $E[B]$ and obtain:

$$E[B](1 - \lambda\mu) = \mu$$

Since we obviously cannot have $E[B] < 0$, this equation only makes sense if $\lambda\mu < 1$, in which case we get:

$$E[B] = \frac{\mu}{1 - \lambda\mu}.$$

Example 5.27 (cont.)

Similarly, by conditioning on S_i we also get:

$$\mathrm{Var}[B_i | S_i] = \mathrm{Var}\left[\sum_{j=1}^{N_u(S_i)} B_{i+j} \,|\, S_i\right] = \lambda S_i E[B^2]$$

Hence, it follows that:

$$\begin{aligned}
\mathrm{Var}[B] &= \mathrm{Var}[E[B_i|S_i]] + E[\mathrm{Var}[B_i|S_i]] \\
&= \mathrm{Var}[(1 + \lambda E[B])S_i] + E[\lambda S_i E[B^2]] \\
&= (1 + \lambda E[B])^2 \,\mathrm{Var}[S_i] + \lambda E[S_i] E[B^2] \\
&= (1 + \lambda E[B])^2 \sigma^2 + \lambda \mu E[B^2] \\
&= (1 + \lambda E[B])^2 \sigma^2 + \lambda \mu (\mathrm{Var}[B] + (E[B])^2)
\end{aligned}$$

Example 5.27 (cont.)

Thus, we have arrived at the following equation:

$$\mathrm{Var}[B] = (1 + \lambda E[B])^2\sigma^2 + \lambda\mu(\mathrm{Var}[B] + (E[B])^2),$$

which we solve with respect to $\mathrm{Var}[B]$ and obtain:

$$\mathrm{Var}[B] = \frac{(1 + \lambda E[B])^2\sigma^2 + \lambda\mu(E[B])^2}{1 - \lambda\mu}$$

By inserting that $E[B] = \mu/(1 - \lambda\mu)$, and simplifying we eventually get:

$$\mathrm{Var}[B] = \frac{\sigma^2 + \lambda\mu^3}{(1 - \lambda\mu)^3}$$

Example 5.27 (cont.)

NOTE: When calculating $E[B]$ and $\mathrm{Var}[B]$, we made the assumption that:

$$\lambda\mu < 1$$

This condition is equivalent to:

$$\mu < 1/\lambda$$

Thus, for the solutions to be valid, the expected service time must be less than the expected time between arrivals.

If $\mu \geq 1/\lambda$, the clients will arrive too frequently compared to the average service time (on average), and as a result the size of the queue will eventually explode.

Under such circumstances the B_is will not have a stable distribution. Instead the B_is will tend to get higher and higher as i grows.

Compound Poisson Process (cont.)

As before, we let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ, and let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed variables, and independent of $\{N(t) : t \geq 0\}$.

Finally, let $\{X(t) : t \geq 0\}$ be the resulting compound Poisson process. That is:

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

We now consider the special case where:

$$P(Y_i = y_j) = p_j, \quad j \in \mathcal{Y},$$

where the set $\mathcal{Y}$ is finite or countably infinite, and $\sum_{j \in \mathcal{Y}} p_j = 1$.

Compound Poisson Process (cont.)

We then let:

$$N_j(t) = \text{The number of events in } [0, t] \text{ where } Y_i = y_j, \quad j \in \mathcal{Y}.$$

Then it follows from previous results that $\{N_j(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λp_j. Moreover, the processes are independent of each other.

Hence it also follows that for any given $t > 0$, $N_1(t), N_2(t), \ldots$ are independent Poisson variables, and that:

$$E[N_j(t)] = \lambda p_j t, \quad t \geq 0, \quad j \in \mathcal{Y}.$$

Compound Poisson Process (cont.)

Moreover, it follows that we have:

$$X(t) = \sum_{j \in \mathcal{Y}} y_j N_j(t).$$

Hence, we get that:

$$E[X(t)] = E\left[\sum_{j \in \mathcal{Y}} y_j N_j(t)\right] = \sum_{j \in \mathcal{Y}} y_j E[N_j(t)]$$

$$= \sum_{j \in \mathcal{Y}} y_j \lambda p_j t \;=\; \lambda t \cdot \sum_{j \in \mathcal{Y}} y_j P(Y_i = y_j) \;=\; \lambda t \cdot E[Y_i],$$

as before.

Compound Poisson Process (cont.)

Similarly, by using that $X(t) = \sum_{j \in \mathcal{Y}} y_j N_j(t)$ we also get:

$$\begin{aligned}
\mathrm{Var}[X(t)] &= \mathrm{Var}\left[\sum_{j \in \mathcal{Y}} y_j N_j(t)\right] \\
&= \sum_{j \in \mathcal{Y}} y_j^2 \,\mathrm{Var}[N_j(t)] \qquad \text{by the independence of the } N_j(t)\text{s} \\
&= \sum_{j \in \mathcal{Y}} y_j^2 \lambda p_j t \;=\; \lambda t \cdot \sum_{j \in \mathcal{Y}} y_j^2 P(Y_i = y_j) \;=\; \lambda t \cdot E[Y_i^2],
\end{aligned}$$

as before.

Compound Poisson Process (cont.)

If $Z \sim Po(\mu)$, it can be shown that $Z \approx N(\mu, \mu)$ provided that the expected value, μ is large.

Hence, by using the above representation for the compound Poisson process $\{X(t) : t \geq 0\}$, it follows that when t is large, we have:

$$X(t) = \sum_{j \in \mathcal{Y}} y_j N_j(t) \approx N(\lambda t \mu, \lambda t \nu)$$

where $\mu = E[Y_i]$ and $\nu = E[Y_i^2]$.

Example 5.28 – Normal approximation

From Example 5.26 we recall that $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate $\lambda = 2$ per week, where:

$$N(t) = \text{Number of families migrating in } [0, t], \quad t \geq 0.$$

$$Y_i = \text{Number of people in the } i\text{th family}, \quad i = 1, 2, \ldots$$

Moreover, we calculated that $E[Y_i] = \frac{5}{2}$, and $E[Y_i^2] = \frac{43}{6}$.

We want to calculate the approximate probability that at least 240 people migrate within the next 50 weeks.

$$E[X(50)] = \lambda t E[Y_i] = 2 \cdot 50 \cdot \tfrac{5}{2} = 250,$$

$$\mathrm{Var}[X(50)] = \lambda t \cdot E[Y_i^2] = 2 \cdot 50 \cdot \tfrac{43}{6} = \tfrac{4300}{6}.$$

Example 5.28 – Normal approximation

Using the so-called continuity correction we then have:

$$P(X(50) \geq 240) = P(X(50) > 239) \approx P(U \geq 239.5)$$

where $U \sim N(250, \frac{4300}{6})$.

By using this we get:

$$P(X(50) \geq 240) \approx P(U \geq 239.5) = P\left(\frac{U - 250}{\sqrt{4300/6}} \geq \frac{239.5 - 250}{\sqrt{4300/6}}\right)$$

$$= 1 - \phi(-0.3922) = 0.6525$$

Sums of compound Poisson Processes

Let $\{X_i(t) : t \geq 0\}$ be a compound Poisson process with rate λ_i, and where the random variables associated with the events have a cumulative distribution function G_i, $i = 1, \ldots, n$.

We assume that the processes $\{X_1(t) : t \geq 0\}, \ldots, \{X_n(t) : t \geq 0\}$ are independent, and let:

$$X(t) = \sum_{i=1}^{n} X_i(t), \quad t \geq 0.$$

Then $\{X(t) : t \geq 0\}$ is also a compound Poisson process with rate:

$$\lambda = \sum_{i=1}^{n} \lambda_i$$

and where the random variables associated with the events have a cumulative distribution function:

$$G(y) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} G_i(y).$$

Sums of compound Poisson Processes (cont.)

To explain why this is true, we first let $\{N_i(t) : t \geq 0\}$ denote the Poisson process generating the events of the compound process $\{X_i(t) : t \geq 0\}$, $i = 1, \ldots, n$.

We then let $\{N(t) : t \geq 0\}$ denote the process generating the events of the process $\{X(t) : t \geq 0\}$. Then we must have:

$$N(t) = \sum_{i=1}^{n} N_i(t), \quad t \geq 0.$$

Since we know that:

$$N_i(t) \sim Po(\lambda t), \quad t \geq 0, \quad i = 1, \ldots, n,$$

and a sum of independent Poisson variables is a Poisson variable with rate equal to the sum of the independent variables, it follows that:

$$N(t) \sim Po(\lambda_1 t + \cdots + \lambda_n t), \quad t \geq 0.$$

Sums of compound Poisson Processes (cont.)

By extending this argument, we may verify the axioms and show that $\{N(t) : t \geq 0\}$ is a Poisson process with rate:

$$\lambda = \sum_{i=1}^{n} \lambda_i.$$

Moreover, it can be shown that for any given event, the probability that it is generated by the Poisson process $\{N_i(t) : t \geq 0\}$ is λ_i/λ, $i = 1, \ldots, n$.

We now consider an arbitrary event with associated random variable Y, and let I denote the index of the Poisson process generating this event. Then by conditioning on I, we have:

$$P(Y \leq y) = \sum_{i=1}^{n} P(Y \leq y | I = i) P(I = i) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} G_i(y) \qquad \blacksquare$$

Sums of compound Poisson Processes (cont.)

EXAMPLE. As an extension of Example 5.26 we assume that $\{N_i(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λ_i, $i = 1, 2$, where:

$$N_i(t) = \text{Number of families from country } i \text{ in } [0, t], \quad t \geq 0$$

$$Y_{ij} = \text{Number of people in the } j\text{th family from country } i, \quad j = 1, 2, \ldots$$

We assume that $\lambda_1 = 2$ per week, and $\lambda_2 = 3$ per week, and that:

$$P(Y_{1j} = 1) = \tfrac{1}{6}, \quad P(Y_{1j} = 2) = \tfrac{2}{6}, \quad P(Y_{1j} = 3) = \tfrac{2}{6}, \quad P(Y_{1j} = 4) = \tfrac{1}{6},$$

$$P(Y_{2j} = 1) = \tfrac{2}{6}, \quad P(Y_{2j} = 2) = \tfrac{2}{6}, \quad P(Y_{2j} = 3) = \tfrac{1}{6}, \quad P(Y_{2j} = 4) = \tfrac{1}{6}$$

Sums of compound Poisson Processes (cont.)

We then let $N(t) = N_1(t) + N_2(t)$, and define:

$$X(t) = \sum_{j=1}^{N(t)} Y_j,$$

where Y_j denotes the number of people in the jth family in the combined process.

Then $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2 = 2 + 3 = 5$.

Moreover, $\{X(t) : t \geq 0\}$ is a compound Poisson process, and we note that:

$$\frac{\lambda_1}{\lambda} = \frac{2}{5}, \qquad \frac{\lambda_2}{\lambda} = \frac{3}{5}.$$

Sums of compound Poisson Processes (cont.)

The distribution of Y_j is given by:

$$P(Y_j = 1) = \tfrac{1}{6} \cdot \tfrac{2}{5} + \tfrac{2}{6} \cdot \tfrac{3}{5} = \tfrac{8}{30}$$

$$P(Y_j = 2) = \tfrac{2}{6} \cdot \tfrac{2}{5} + \tfrac{2}{6} \cdot \tfrac{3}{5} = \tfrac{10}{30}$$

$$P(Y_j = 3) = \tfrac{2}{6} \cdot \tfrac{2}{5} + \tfrac{1}{6} \cdot \tfrac{3}{5} = \tfrac{7}{30}$$

$$P(Y_j = 4) = \tfrac{1}{6} \cdot \tfrac{2}{5} + \tfrac{1}{6} \cdot \tfrac{3}{5} = \tfrac{5}{30}$$

Hence, we get:

$$E[Y_j] = 1 \cdot \tfrac{8}{30} + 2 \cdot \tfrac{10}{30} + 3 \cdot \tfrac{7}{30} + 4 \cdot \tfrac{5}{30} = \tfrac{69}{30}$$

$$E[Y_j^2] = 1^2 \cdot \tfrac{8}{30} + 2^2 \cdot \tfrac{10}{30} + 3^2 \cdot \tfrac{7}{30} + 4^2 \cdot \tfrac{5}{30} = \tfrac{191}{30}$$

Sums of compound Poisson Processes (cont.)

We then have:

$$E[X(3)] = \lambda \cdot 3 \cdot E[Y_i] = 5 \cdot 3 \cdot \tfrac{69}{30} = \tfrac{1035}{30} = 34.5$$

$$\mathrm{Var}[X(3)] = \lambda \cdot 3 \cdot E[Y_i^2] = 5 \cdot 3 \cdot \tfrac{191}{30} = \tfrac{2865}{30} = 95.5.$$