STK2130 – Chapter 5.4 Overview

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The non-homogeneous Poisson Process

Definition

A counting process $\{N(t) : t \geq 0\}$ *is said to be a non-homogeneous Poisson process with intensity function* $\lambda(t)$, $t \geq 0$, *if:*

- *(i)* $N(0) = 0$
- *(ii)* $\{N(t), t \geq 0\}$ *has independent increments.*
- *(iii)* $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$
- *(iv)* $P(N(t+h) - N(t) \geq 2) = o(h)$

We also introduce the mean value function $m(t)$ *defined by:*

$$m(t) = \int_0^t \lambda(u)du$$

The non-homogeneous Poisson Process (cont.)

Theorem (5.3)

If $\{N(t) : t \geq 0\}$ *is a non-homogeneous Poisson process with intensity function* $\lambda(t)$*, then:*

$$P(N(t) = n) = \frac{(m(t))^n}{n!} e^{-m(t)}, \quad t \geq 0, \quad n = 0, 1, 2, \ldots$$

The non-homogeneous Poisson process (cont.)

Let $\{N(t) : t \geq 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$, and mean value function $m(t)$. Furthermore, let:

$$S_n = \text{The time of the } n\text{th event}, \quad n = 1, 2, \ldots$$

The density of $S_1 = T_1$ is given by:

$$f_{S_1}(t) = \lambda(t) e^{-m(t)}, \quad t \geq 0.$$

More generally the density of S_n, $n = 1, 2, \ldots$ is given by:

$$f_{S_n}(t) = \frac{[m(t)]^{n-1}}{(n-1)!} \lambda(t) e^{-m(t)}.$$

The non-homogeneous Poisson process (cont.)

NOTE 1. If $n = 1$, we as before get:

$$f_{S_1}(t) = \frac{[m(t)]^{1-1}}{(1-1)!}\lambda(t)e^{-m(t)} = \lambda(t)e^{-m(t)}.$$

NOTE 2. If $\lambda(t) = \lambda$, then $m(t) = \int_0^t \lambda du = \lambda t$, and we get:

$$f_{S_n}(t) = \frac{[m(t)]^{n-1}}{(n-1)!}\lambda(t)e^{-m(t)}$$

$$= \frac{[\lambda t]^{n-1}}{(n-1)!}\lambda e^{-\lambda t} = \frac{\lambda^n}{\Gamma(n)}t^{n-1}e^{-\lambda t}$$

Thus, in this case $S_n \sim Gamma(n, \lambda)$ as before.

Time sampling a Poisson Process

Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson Process with rate λ where each event can be classified as either a Type 1 event or a Type 2 event.

If an event occurs at time t, then the probability that it is of type 1 is $p_1(t)$, and the probability that it is of type 2 is $p_2(t) = 1 - p_1(t)$.

We assume that the event type at time t is independent of the history of the Poisson process up to time t, and introduce:

$$N_i(t) = \text{The number of events of type } i \text{ in } [0, t] \quad t \geq 0, \quad i = 1, 2.$$

- $\{N_i(t) : t \geq 0\}$ is a non-homogeneous Poisson Process with intensity function $\lambda p_i(t)$, $i = 1, 2$.
- $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent of each other.

Compound Poisson Process

Let $\{N(t) : t \geq 0\}$ be a Poisson process, and let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed variables, and independent of $\{N(t) : t \geq 0\}$.

We then define a new stochastic process $\{X(t) : t \geq 0\}$ such that:

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

The process $\{X(t) : t \geq 0\}$ is said to be a compound Poisson process.

NOTE: If $P(Y_i = 1) = 1$, $i = 1, 2, \ldots$, then obviously $X(t) = N(t)$. Thus, a (regular) Poisson process is a special case of a compound Poisson process.

Compound Poisson Process (cont.)

EXAMPLE 1. An insurance company receives claims from its clients at random points of time. We let:

$$N(t) = \text{The number of claims in } [0, t], \quad t \geq 0$$

and assume that $\{N(t) : t \geq 0\}$ is a Poisson process with rate λ. Moreover, we let:

$$Y_i = \text{The size in NOK of the } i\text{th claim}, \quad i = 1, 2, \ldots,$$

and assume that $Y_1, Y_2, \ldots$ are independent and identically distributed variables and independent of $\{N(t) : t \geq 0\}$.

We then introduce:

$$X(t) = \sum_{i=1}^{N(t)} Y_i = \text{The sum of claims in } [0, t], \quad t \geq 0.$$

Then $\{X(t) : t \geq 0\}$ is a compound Poisson process.

Compound Poisson Process (cont.)

Let $E[Y_i] = \mu$ and $E[Y_i^2] = \nu$, $i = 1, 2, \ldots$. Thus, $\mathrm{Var}[Y_i] = \nu - \mu^2$.

The expectation and variance of $X(t)$, calculated by conditioning on $N(t)$ is:

$$E[X(t)] = E[E[\sum_{i=1}^{n} Y_i | N(t) = n]] \;=\; \lambda t \cdot E[Y_i]$$

$$\mathrm{Var}[X(t)] = \mathrm{Var}[E[\sum_{i=1}^{n} Y_i | N(t) = n]] + E[\mathrm{Var}[\sum_{i=1}^{n} Y_i | N(t) = n]]$$

$$= = \;\lambda t \cdot E[Y_i^2]$$

Compound Poisson Process (cont.)

As before, we let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ, and let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed variables, and independent of $\{N(t) : t \geq 0\}$.

Finally, let $\{X(t) : t \geq 0\}$ be the resulting compound Poisson process. That is:

$$X(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0.$$

We now consider the special case where:

$$P(Y_i = y_j) = p_j, \quad j \in \mathcal{Y},$$

where the set $\mathcal{Y}$ is finite or countably infinite, and $\sum_{j \in \mathcal{Y}} p_j = 1$.

Compound Poisson Process (cont.)

We then let:

$$N_j(t) = \text{The number of events in } [0, t] \text{ where } Y_i = y_j, \quad j \in \mathcal{Y}.$$

Then $\{N_j(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λp_j, $j \in \mathcal{Y}$.

Moreover, the processes are independent of each other.

Finally, we have:

$$X(t) = \sum_{j \in \mathcal{Y}} y_j N_j(t).$$

Compound Poisson Process (cont.)

If $Z \sim Po(\mu)$, it can be shown that $Z \approx N(\mu, \mu)$ provided that the expected value, μ is large.

By using the above representation, it follows that when t is large, we have:

$$X(t) = \sum_{j \in \mathcal{Y}} y_j N_j(t) \approx N(\lambda t E[Y_i], \lambda t E[Y_i^2])$$

Sums of compound Poisson Processes

Let $\{X_i(t) : t \geq 0\}$ be a compound Poisson process with rate λ_i, and where the random variables associated with the events have a cumulative distribution function G_i, $i = 1, \ldots, n$.

We assume that the processes $\{X_1(t) : t \geq 0\}, \ldots, \{X_n(t) : t \geq 0\}$ are independent, and let:

$$X(t) = \sum_{i=1}^{n} X_i(t), \quad t \geq 0.$$

Then $\{X(t) : t \geq 0\}$ is also a compound Poisson process with rate:

$$\lambda = \sum_{i=1}^{n} \lambda_i$$

and where the random variables associated with the events have a cumulative distribution function:

$$G(y) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} G_i(y).$$

Remaining lectures - Chapter 6

- Lecture 1. (Week 14)
  - Chapter 6.2 Continuous-Time Markov Chains
  - Chapter 6.3 Birth and Death Processes
- Lecture 2. (Week 16)
  - Chapter 6.4 The Transition Probability Function $P_{ij}(t)$
- Lecture 3. (Week 17)
  - Chapter 6.5 Limiting Probabilities
  - Chapter 6.8 Uniformization
  - Chapter 6.9 Computing the Transition Probabilities

Remaining lectures - Chapter 7

- Lecture 4. (Week 18)
  - Chapter 7.1 Renewal Theory and Its Applications
  - Chapter 7.2 Distribution of $N(t)$

Remaining lectures - Chapter 10

- Lecture 5. (Week 19)
  - Chapter 10.1 Brownian Motion
  - Chapter 10.2 Hitting Times, Maximum Variable, and the Gambler's Ruin Problem
  - Chapter 10.3 Variations on Brownian Motion

EXAM: (Week 22) May 27, 14:30 – June 3, 14:30