STK2130 – Chapter 6.2

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Discrete-time Markov Chains

We recall from Chapter 4:

Let $\{X_n : n \geq 0\}$ be a discrete-time stochastic process with discrete state space $\mathcal{X}$.

The process is a Markov chain if for $n = 1, 2, \ldots$ we have:

$$P(X_{n+1} = j | X_n = i, X_u = x_u, 0 \leq u < n)$$

$$= P(X_{n+1} = j | X_n = i), \quad i, j, x_u \in \mathcal{X}$$

If we also have that $P(X_{n+1} = j | X_n = i)$ is independent of n, then the Markov chain is said to have stationary (or homogeneous) transition probabilities.

6.2 Continuous-Time Markov Chains

Let $\{X(t) : t \geq 0\}$ be a continuous-time stochastic process with discrete state space $\mathcal{X}$.

The process is a Markov chain if for $s, t > 0$ we have:

$$P(X(t+s) = j | X(s) = i,\ X(u) = x(u),\ 0 \leq u < s)$$

$$= P(X(t+s) = j | X(s) = i), \quad i, j, x(u) \in \mathcal{X}$$

If we also have that $P(X(t+s) = j | X(s) = i)$ is independent of s, then the Markov chain is said to have stationary (or homogeneous) transition probabilities.

6.2 Continuous-Time Markov Chains (cont.)

EXAMPLE: Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ. This process has independent and stationary increments.

Hence, for $j \geq i$ and $s, t > 0$ we have:

$$P(N(t+s) = j | N(s) = i, N(u) = n(u), 0 \leq u < s)$$

$$= P(N(t+s) = j | N(s) = i) = P(N(t+s) - N(s) = j - i)$$

$$= \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}, \quad \text{independent of } s$$

For $j < i$ the corresponding probabilities are zero.

Hence, $\{N(t) : t \geq 0\}$ is a Markov chain.

6.2 Continuous-Time Markov Chains (cont.)

Assume that $X(0) = i$, and define:

$$T_i = \inf\{u \geq 0 : X(u) \neq i\}$$

Thus, T_i is the point of time when the process leaves state i.

We then let $s, t > 0$, and consider:

$$\begin{aligned}
&P(T_i > s + t | T_i > s) \\
&\qquad = P(X(u) = i, 0 \leq u \leq s + t | X(u) = i, 0 \leq u \leq s) \\
&\qquad = P(X(u) = i, s \leq u \leq s + t | X(s) = i), \quad \text{by the Markov property} \\
&\qquad = P(X(u) = i, 0 \leq u \leq t | X(0) = i), \quad \text{by the stationary property} \\
&\qquad = P(T_i > t).
\end{aligned}$$

This implies that T_i is memoryless, and hence T_i is exponentially distributed.

6.2 Continuous-Time Markov Chains (cont.)

Assume more generally that $X(r) = i$, and define:

$$T_i = \inf\{u \geq 0 : X(r+u) \neq i\}$$

Thus, $T_i + r$ is the point of time when the process leaves state i.

We then let $s, t > 0$, and consider:

$$\begin{aligned}
&P(T_i > s + t | T_i > s) \\
&\qquad = P(X(u) = i, r \leq u \leq r + s + t | X(u) = i, r \leq u \leq r + s) \\
&\qquad = P(X(u) = i, r + s \leq u \leq r + s + t | X(r+s) = i), \quad \text{by Markov} \\
&\qquad = P(X(u) = i, r \leq u \leq r + t | X(r) = i), \quad \text{by stationarity} \\
&\qquad = P(T_i > t).
\end{aligned}$$

This implies that T_i is memoryless, and hence T_i is exponentially distributed.

6.2 Continuous-Time Markov Chains (cont.)

ALTERNATIVE DEFINITION:
A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables (because of the Markov property).
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with some mean v_i^{-1} (because of the Markov property and stationarity).
- When the process leaves state i, it enters state j with some transition probability Q_{ij} where:

$$Q_{ii} = 0, \quad \text{for all } i \in \mathcal{X}$$

$$\sum_{j \in \mathcal{X}} Q_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.

Example 6.1 – A Shoe Shine Shop

A Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2\}$ where:

- State 0. No customer
- State 1. Customer in chair 1 (clean and polish)
- State 2. Customer in chair 2 (polish is buffed)

$X(s) = 0$: In this state customers arrive in accordance to a Poisson process with rate λ. The time spent in this state is $T_0 \sim exp(\lambda)$. Then the process transits to state 1 with probability $Q_{01} = 1$.

$X(t) = 1$: The time spent in this state is $T_1 \sim exp(\mu_1)$. Then the process transits to state 2 with probability $Q_{12} = 1$.

$X(u) = 2$: The time spent in this state is $T_2 \sim exp(\mu_2)$. Then the process transits to state 0 with probability $Q_{20} = 1$, and then the process repeats the same cycle.

Example 6.1 (cont.)

Thus, the transition probability matrix of the built-in discrete time Markov chain is:

$$\boldsymbol{Q} = \begin{bmatrix} 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}$$

Thus, the built-in discrete time Markov chain is periodic with a period length of 3.

NOTE: Even though the built-in discrete time Markov chain is periodic, the continuous-time Markov chain $\{X(t) : t \geq 0\}$ will have a well-defined limiting distribution.

Example: A multistate component

A Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2\}$ where:

- State 0. The component is failed
- State 1. The component is functioning but not perfectly
- State 2. The component is functioning perfectly

$X(s) = 2$: The time spent in this state is $T_2 \sim exp(\mu_2)$. Then the process transits to state 1 with probability $Q_{21} = 0.5$ or to state 0 with probability $Q_{20} = 0.5$.

$X(t) = 1$: The time spent in this state is $T_1 \sim exp(\mu_1)$. Then the process transits to state 0 with probability $Q_{10} = 1$.

$X(u) = 0$: The time spent in this state is $T_0 \sim exp(\mu_0)$. Then the component is repaired and the process transits to state 2 with probability $Q_{02} = 1$, and then the process repeats the same cycle.

Example: A multistate component (cont.)

Thus, the transition probability matrix of the built-in discrete time Markov chain is:

$$\boldsymbol{Q} = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 0.0 \\ 0.5 & 0.5 & 0.0 \end{bmatrix}$$

In this case the built-in discrete time Markov chain is aperiodic, and the limiting distribution, $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2)$, found by solving:

$$\boldsymbol{\pi Q} = \boldsymbol{\pi}$$

$$\boldsymbol{\pi 1} = 1$$

is given by:

$$\pi_0 = 0.4, \qquad \pi_1 = 0.2, \qquad \pi_2 = 0.4$$