STK2130 – Chapter 6.3

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6.3 Birth and Death Processes

A Birth and Death Process $\{X(t) : t \geq 0\}$ has state space $\mathcal{X} = \{0, 1, 2, \ldots\}$.

Assume that $X(t) = n > 0$. Then the next transition is determined as follows:

- Sample $V \sim exp(\lambda_n)$ and $W \sim exp(\mu_n)$ independent of each other with respective outcomes v and w.
- If $v < w$ then the process transits to state $n + 1$ at time $t + v$, i.e., $X(t + v) = n + 1$. This called a birth.
- If $w < v$ then the process transits to state $n - 1$ at time $t + w$, i.e., $X(t + w) = n - 1$. This called a death.

NOTE: When $X(t) = 0$, only births are possible, so in this case we assume that $W = \infty$, which corresponds to the rate μ_0 being zero, and $P_{01} = 1$.

6.3 Birth and Death Processes (cont.)

The transition (either a birth or a death) happens at time $U = \min(V, W)$. Hence, the distribution of U can be derived as follows:

$$
\begin{aligned}
P(U > u) &= P(V > u \cap W > u) \\
&= P(V > u) \cdot P(W > u) \quad \text{since } V \text{ and } W \text{ are independent} \\
&= e^{-(\lambda_n)u} \cdot e^{-(\mu_n)u} \quad \text{since } V \sim exp(\lambda_n) \text{ and } W \sim exp(\mu_n) \\
&= e^{-(\lambda_n + \mu_n)u}
\end{aligned}
$$

Hence, it follows that $U \sim exp(\lambda_n + \mu_n)$.

6.3 Birth and Death Processes (cont.)

The transition probabilities for the built-in discrete-time Markov chain can be derived as follows:

$$
\begin{aligned}
P_{n,n+1} = P(V < W) &= \int_0^\infty P(V < W | V = v)\lambda_n e^{-\lambda_n v} dv \\
&= \int_0^\infty e^{-\mu_n v}\lambda_n e^{-\lambda_n v} dv \\
&= \frac{\lambda_n}{\lambda_n + \mu_n}\int_0^\infty (\lambda_n + \mu_n) e^{-(\lambda_n+\mu_n)v} dv \\
&= \frac{\lambda_n}{\lambda_n + \mu_n}
\end{aligned}
$$

Hence, we also get that:

$$
P_{n,n-1} = P(V > W) = 1 - P(V < W) = 1 - \frac{\lambda_n}{\lambda_n + \mu_n} = \frac{\mu_n}{\lambda_n + \mu_n}
$$

Example 6.2 – A Pure Birth Process

Assume that $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\begin{aligned} \mu_n &= 0, \quad \text{for all } n \geq 0 \\ \lambda_n &= \lambda, \quad \text{for all } n \geq 0 \end{aligned}$$

Since the death rate is zero, this is a pure birth process with constant birth rate λ.

This implies that the time between transitions is exponentially distributed with rate λ.

Hence, $\{X(t) : t \geq 0\}$ is a Poisson process with rate λ.

Example 6.3 – The Yule Process

Assume that $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\mu_n = 0, \quad \text{for all } n \geq 0$$
$$\lambda_n = \lambda n, \quad \text{for all } n \geq 0$$

Since the death rate is zero, this is a pure birth process. The birth rate λn is proportional to the state, i.e., number of individuals in the population.

This implies that the time the process stays in state n is exponentially distributed with rate λn. Thus, the expected time between transitions becomes smaller and smaller as n grows.

Example 6.5 – An $M/M/1$-queue

An $M/M/1$-queue is a queue where:

- **M**arkov arrival process: The times between arrivals are independent and exponentially distributed with rate λ.
- **M**arkov service process: The service times are independent and exponentially distributed with rate μ.
- **1** server: The maximal number of customers that can be served at a time is 1
- $X(t)$ be the number of customers in the queue at time t.

Then $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\mu_n = \mu \cdot \min(n, 1), \quad \text{for all } n \geq 0$$
$$\lambda_n = \lambda, \quad \text{for all } n \geq 0$$

Example 6.6 – An $M/M/s$-queue

An $M/M/s$-queue is a queue where:

- **M**arkov arrival process: The times between arrivals are independent and exponentially distributed with rate λ.
- **M**arkov service process: The service times are independent and exponentially distributed with rate μ.
- **s** server: The maximal number of customers that can be served at a time is s
- $X(t)$ be the number of customers in the queue at time t.

Then $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\mu_n = \mu \cdot \min(n, s), \quad \text{for all } n \geq 0$$
$$\lambda_n = \lambda, \quad \text{for all } n \geq 0$$

Example 6.4 – Linear Growth with Immigration

Assume that $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\mu_n = \mu n, \quad \text{for all } n \geq 1$$
$$\lambda_n = \lambda n + \theta, \quad \text{for all } n \geq 0$$

Each member of the population gives birth with a rate λ.

In addition the population also increases due to immigration (independent of the births in the population) at a rate of θ.

Deaths occur at a rate of μ for each member of the population.

Example 6.4 – Linear Growth with Immigration (cont.)

We assume that $X(0) = i$ and introduce:

$$M(t) = E[X(t)]$$

We will determine $M(t)$ by solving a differential equation, and start by establishing the following:

$$M(t+h) = E[X(t+h)] = E[E[X(t+h)|X(t)]]$$

Since the time between transitions is exponentially distributed, the probability of more than one transition in an interval of length h is $o(h)$. Hence, we have:

$$P(X(t+h) = X(t)+1|X(t)) = [X(t)\lambda + \theta]h + o(h)$$

$$P(X(t+h) = X(t)-1|X(t)) = X(t)\mu h + o(h)$$

$$P(X(t+h) = X(t)|X(t)) = 1 - [X(t)\lambda + \theta + X(t)\mu]h + o(h)$$

Example 6.4 – Linear Growth with Immigration (cont.)

From this it follows that:

$$\begin{aligned} E[X(t+h)|X(t)] &= X(t) + [X(t)\lambda + \theta]h - X(t)\mu h + o(h) \\ &= X(t) + (\lambda - \mu)X(t)h + \theta h + o(h) \end{aligned}$$

Hence, by taking expectations on both sides, we get:

$$M(t+h) = M(t) + (\lambda - \mu)M(t)h + \theta h + o(h)$$

and thus:

$$\frac{M(t+h) - M(t)}{h} = (\lambda - \mu)M(t) + \theta + \frac{o(h)}{h}$$

By taking the limit as $h \to 0$, we obtain the following differential equation:

$$M'(t) = (\lambda - \mu)M(t) + \theta$$

Example 6.4 – Linear Growth with Immigration (cont.)

We rewrite the equation as:

$$M'(t) - (\lambda - \mu)M(t) = \theta$$

Assuming that $\lambda \neq \mu$, we can solve this by multiplying both sides by the integrating factor $e^{-(\lambda-\mu)t}$:

$$M'(t)e^{-(\lambda-\mu)t} - (\lambda - \mu)e^{-(\lambda-\mu)t}M(t) = \theta e^{-(\lambda-\mu)t}$$

This equation can be expressed as:

$$[M(t) \cdot e^{-(\lambda-\mu)t}]' = \theta e^{-(\lambda-\mu)t}$$

Integrating both sides yields:

$$M(t) \cdot e^{-(\lambda-\mu)t} = -\frac{\theta}{\lambda - \mu} e^{-(\lambda-\mu)t} + C$$

Example 6.4 – Linear Growth with Immigration (cont.)

Hence, by multiplying both sides by $e^{(\lambda-\mu)t}$ we get:

$$M(t) = -\frac{\theta}{\lambda - \mu} + Ce^{(\lambda-\mu)t}$$

In order to determine the constant C, we use the boundary condition that $X(0) = i$, which also implies that $M(0) = E[X(0)] = i$. By inserting this we get:

$$i = -\frac{\theta}{\lambda - \mu} + C,$$

which implies that:

$$C = \frac{\theta}{\lambda - \mu} + i$$

Example 6.4 – Linear Growth with Immigration (cont.)

By inserting this into the expression for $M(t)$, we get:

$$M(t) = -\frac{\theta}{\lambda - \mu} + [\frac{\theta}{\lambda - \mu} + i]e^{(\lambda-\mu)t}$$
$$= \frac{\theta}{\lambda - \mu}[e^{(\lambda-\mu)t} - 1] + ie^{(\lambda-\mu)t}$$

For the case where $\lambda = \mu$ the differential equation:

$$M'(t) - (\lambda - \mu)M(t) = \theta$$

simplifies to $M'(t) = \theta$, which have the solution:

$$M(t) = \theta t + i$$

6.3 Birth and Death Processes (cont.)

We consider a general birth and death process, $\{X(t) : t \geq 0\}$, with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$, where $\mu_0 = 0$.

Assume that $X(0) = i$, where $i \geq 0$, and define T_i to be the time until the process enters state $i + 1$ for the first time.

GOAL: Calculate $E[T_i]$.

Since $T_0 \sim exp(\lambda_0)$, we know that:

$$E[T_0] = \frac{1}{\lambda_0}.$$

6.3 Birth and Death Processes (cont.)

We then introduce:

$$I_i = \begin{cases} 1 & \text{if the first transition from } i \text{ is to } i+1 \\ 0 & \text{if the first transition from } i \text{ is to } i-1 \end{cases}$$

By conditioning on I_i being either 1 or 0, and using that the expected time until the first transition is $(\lambda_i + \mu_i)^{-1}$, we get:

$$E[T_i | I_i = 1] = \frac{1}{\lambda_i + \mu_i},$$

$$E[T_i | I_i = 0] = \frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i].$$

The unconditional expectation then becomes:

$$E[T_i] = \frac{1}{\lambda_i + \mu_i} \cdot P(I_i = 1) + \left(\frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i] \right) \cdot P(I_i = 0)$$

6.3 Birth and Death Processes (cont.)

Furthermore, we have that:

$$P(I_i = 1) = \frac{\lambda_i}{\lambda_i + \mu_i}$$

$$P(I_i = 0) = \frac{\mu_i}{\lambda_i + \mu_i}$$

Hence, we get that:

$$E[T_i] = \frac{1}{\lambda_i + \mu_i} \cdot P(I_i = 1) \ + \ \left(\frac{1}{\lambda_i + \mu_i} + E[T_{i-1}] + E[T_i] \right) \cdot P(I_i = 0)$$

$$= \frac{1}{\lambda_i + \mu_i} [P(I_i = 1) + P(I_i = 0)] \ + \ P(I_i = 0)[E[T_{i-1}] + E[T_i]]$$

$$= \frac{1}{\lambda_i + \mu_i} \ + \ \frac{\mu_i}{\lambda_i + \mu_i} [E[T_{i-1}] + E[T_i]]$$

6.3 Birth and Death Processes (cont.)

This equation:

$$E[T_i] = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i}[E[T_{i-1}] + E[T_i]]$$

can alternatively be written as:

$$E[T_i](1 - \frac{\mu_i}{\lambda_i + \mu_i}) = E[T_i](\frac{\lambda_i}{\lambda_i + \mu_i}) = \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i}E[T_{i-1}]$$

We then multiply both sides of the equation by $(\lambda_i + \mu_i)$, and get:

$$E[T_i]\lambda_i = 1 + \mu_i E[T_{i-1}]$$

Finally, we divide both sides by λ_i and get:

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}E[T_{i-1}]$$

6.3 Birth and Death Processes (cont.)

By using this recursive relation, and that $E[T_0] = \lambda_0^{-1}$, we get:

$$E[T_0] = \frac{1}{\lambda_0}$$

$$E[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}$$

$$E[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}[\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}]$$

$$\cdots$$

6.3 Birth and Death Processes (cont.)

Assume more specifically that $\lambda_i = \lambda$, $i = 0, 1, 2, \ldots$, and $\mu_i = \mu$, $i = 1, 2, 3, \ldots$. Then we have:

$$E[T_0] = \frac{1}{\lambda}$$

$$E[T_1] = \frac{1}{\lambda} + \frac{\mu}{\lambda}\frac{1}{\lambda} = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda}]$$

$$E[T_2] = \frac{1}{\lambda} + \frac{\mu}{\lambda}[\frac{1}{\lambda} + \frac{\mu}{\lambda}\frac{1}{\lambda}] = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2]$$

$$\cdots$$

$$E[T_i] = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \cdots + \left(\frac{\mu}{\lambda}\right)^i]$$

6.3 Birth and Death Processes (cont.)

In the case where $\lambda \neq \mu$, we can use the formula for the sum of a geometric series and obtain:

$$E[T_i] = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \cdots + \left(\frac{\mu}{\lambda}\right)^i]$$

$$= \frac{1}{\lambda} \cdot \frac{(\mu/\lambda)^{i+1} - 1}{(\mu/\lambda) - 1} = \frac{1 - (\mu/\lambda)^{i+1}}{\lambda - \mu}, \quad i = 0, 1, 2, \ldots$$

NOTE: If $i = 0$, we get $E[T_i] = \frac{1-\mu/\lambda}{\lambda-\mu} = \frac{\lambda-\mu}{\lambda(\lambda-\mu)} = \frac{1}{\lambda} = E[T_0]$ as before.

In the case where $\lambda = \mu$, the formula can be simplified as follows:

$$E[T_i] = \frac{1}{\lambda}[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \cdots + \left(\frac{\mu}{\lambda}\right)^i] = \frac{i+1}{\lambda}$$

NOTE: If $i = 0$, we get $E[T_i] = \frac{1}{\lambda} = E[T_0]$ as before.

6.3 Birth and Death Processes (cont.)

More generally, assuming that $X(0) = i$, we let T_{ij} be the time until the process enters state j for the first time, where $j > i$. Then we have:

$$E[T_{ij}] = E[T_{i,i+1}] + E[T_{i+1,i+2}] + \cdots + E[T_{j-1,j}]$$
$$= \sum_{k=i}^{j-1} E[T_k]$$

If $\lambda \neq \mu$ it can be shown that:

$$E[T_{ij}] = \frac{j-i}{\lambda-\mu} - \frac{(\mu/\lambda)^{i+1}}{\lambda-\mu} \frac{1-(\mu/\lambda)^{j-i}}{1-\mu/\lambda}$$

If $\lambda = \mu$ it can be shown that:

$$E[T_{ij}] = \frac{j(j+1) - i(i+1)}{2\lambda}.$$

6.3 Birth and Death Processes (cont.)

We close this by verifying that these last expressions simplifies to the previous expressions when $j = i + 1$:

In the case where $\lambda \neq \mu$ we insert $j = i + 1$ and get:

$$E[T_{i,i+1}] = \frac{j-i}{\lambda - \mu} - \frac{(\mu/\lambda)^{i+1}}{\lambda - \mu} \cdot \frac{1 - (\mu/\lambda)^{j-i}}{1 - \mu/\lambda}$$

$$= \frac{i+1-i}{\lambda - \mu} - \frac{(\mu/\lambda)^{i+1}}{\lambda - \mu} \cdot \frac{1 - (\mu/\lambda)^{i+1-i}}{1 - \mu/\lambda}$$

$$= \frac{1}{\lambda - \mu} - \frac{(\mu/\lambda)^{i+1}}{\lambda - \mu} \cdot \frac{1 - \mu/\lambda}{1 - \mu/\lambda}$$

$$= \frac{1 - (\mu/\lambda)^{i+1}}{\lambda - \mu} = E[T_i]$$

6.3 Birth and Death Processes (cont.)

In the case where $\lambda = \mu$ we again insert $j = i + 1$ and get:

$$\begin{aligned}
E[T_{i,i+1}] &= \frac{j(j+1) - i(i+1)}{2\lambda} \\
&= \frac{(i+1)(i+2) - i(i+1)}{2\lambda} \\
&= \frac{(i+1)(i+2-i)}{2\lambda} \\
&= \frac{i+1}{\lambda} = E[T_i]
\end{aligned}$$