STK2130 – Chapter 6.4 (part 1)

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The hypoexponential distribution (REPRISE)

We recall that if $X \sim exp(\lambda)$, then the moment generating function of X is given by:

$$M_X(t) = E[e^{tX}] = \int_0^\infty \lambda e^{-(\lambda - t)x} dx = \frac{\lambda}{\lambda - t}.$$

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Now, let $X_1, \ldots, X_n$ be independent and $X_i \sim exp(\lambda_i)$, $i = 1, \ldots, n$, and assume that all the λ_i's are distinct. That is $\lambda_i \neq \lambda_j$ for all $i \neq j$.

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The moment generating function of $S = X_1 + \cdots + X_n$ is given by:

$$\begin{aligned} M_S(t) &= E[e^{tS}] = E[e^{tX_1 + \cdots tX_n}] \\ &= E[e^{tX_1}] \cdots E[e^{tX_n}] \quad \text{(since } X_1, \ldots, X_n \text{ are independent)} \\ &= \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \frac{\lambda_i}{\lambda_i - t}. \end{aligned}$$

The hypoexponential distribution (cont.)

Assume that $\lambda_1, \ldots, \lambda_n$ be distinct positive numbers. A random variable Z is said to have a hypoexponential distribution with rates $\lambda_1, \ldots, \lambda_n$ if the density of Z is given by:

$$f_Z(z) = \sum_{i=1}^{n} C_i \cdot \lambda_i e^{-\lambda_i z}, \quad z \geq 0,$$

where:

$$C_i = \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}, \quad i = 1, \ldots, n.$$

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The moment generating function of Z is then given by:

$$M_Z(t) = E[e^{tZ}] = \int_0^\infty \sum_{i=1}^{n} C_i \cdot \lambda_i e^{-(\lambda_i - t)z} dz$$

$$= \sum_{i=1}^{n} C_i \int_0^\infty \lambda_i e^{-(\lambda_i - t)z} dz = \sum_{i=1}^{n} C_i \cdot \frac{\lambda_i}{\lambda_i - t}$$

The hypoexponential distribution (cont.)

By inserting the expressions for $C_1, \ldots, C_n$, we get:

$$M_Z(t) = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t} \cdot C_i \;=\; \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$$

$$= \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t} \cdot \sum_{i=1}^{n} \prod_{j \neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i} \;=\; \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t} \cdot \phi_n(t),$$

where:

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where:

$$\phi_n(t) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i}$$

We observe that $\phi_n(t)$ is a polynomial in t of degree ν, where $\nu \leq (n-1)$.

If $\nu > 0$, the equation $\phi_n(t) = 1$ can have at most $\nu < n$ distinct real solutions.

The hypoexponential distribution (cont.)

However, for $k = 1, \ldots, n$ we must have:

$$\prod_{j \neq i} \frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_i} = 0, \quad \text{if } k \neq i,$$

$$\prod_{j \neq i} \frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_i} = 1, \quad \text{if } k = i.$$

Hence, we get that:

$$\phi_n(\lambda_k) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\lambda_j - \lambda_k}{\lambda_j - \lambda_i} = 1, \quad k = 1, \ldots, n.$$

The hypoexponential distribution (cont.)

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Since we have assumed that $\lambda_1, \ldots, \lambda_n$ are distinct, the equation $\phi_n(t) = 1$ has n distinct real solutions, which implies that $\nu = 0$, i.e., that $\phi_n(t) \equiv 1$.

The hypoexponential distribution (cont.)

Thus, we have shown that the moment generating function of Z is simply:

$$M_Z(t) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i - t} = M_S(t).$$

Since the moment generating function (when it exists) uniquely determines the distribution, this implies that Z has the distribution of a sum of n independent, exponentially distributed variables with distinct rates.

6.4 The Transition Probability Function $P_{ij}(t)$

The transition probabilities of a stationary continuous-time Markov chain $\{X(t) : t \geq 0\}$, with state space $\mathcal{X}$ are defined as:

$$P_{ij}(t) = P(X(t+s) = j | X(s) = i), \quad t \geq 0, \quad i, j \in \mathcal{X}$$

We now consider the case where $\{X(t) : t \geq 0\}$ is a pure birth process, and assume that $X(0) = i$. We then let:

$$T_k = \text{The time spent in state } k, \quad k = i, (i+1), \ldots,$$

and note that if $j > i$, then:

$$X(t) < j \quad \Leftrightarrow \quad T_i + T_{i+1} + \cdots + T_{j-1} > t$$

6.4 Transition Probabilities (cont.)

Hence, for $i < j$ we have:

$$P(X(t) < j | X(0) = i) = P(Z > t)$$

where we have introduced $Z = \sum_{k=i}^{j-1} T_k$.

Assuming that $\lambda_i \neq \lambda_j$ for all $i \neq j$, we know that Z is hypoexponentially distributed, and thus, the density of Z is:

$$f_Z(z) = \sum_{k=i}^{j-1} C_k \cdot \lambda_k e^{-\lambda_k z}, \quad z \geq 0,$$

where:

$$C_k = \prod_{r \neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}, \quad k = i, \ldots, j-1.$$

6.4 Transition Probabilities (cont.)

Hence, we get:

$$P(X(t) < j | X(0) = i) = P(Z > t) = \int_t^{\infty} f_Z(z)dz$$

$$= \sum_{k=i}^{j-1} C_k \cdot \int_t^{\infty} \lambda_k e^{-\lambda_k z} dz$$

$$= \sum_{k=i}^{j-1} C_k \cdot e^{-\lambda_k t}$$

$$= \sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

6.4 Transition Probabilities (cont.)

By exactly the same argument we also get that:

$$P(X(t) < j+1 | X(0) = i) = \sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k}$$

From this we can find:

$$\begin{aligned} P_{ij}(t) &= P(X(t) = j | X(0) = i) \\ &= P(X(t) < j+1 | X(0) = i) - P(X(t) < j | X(0) = i) \end{aligned}$$

$$P_{ii}(t) = P(T_i > t) = e^{-\lambda_i t}$$

The next proposition summarizes these results.

6.4 Transition Probabilities (cont.)

Proposition (6.1)

For a pure birth process where $\lambda_i \neq \lambda_j$ for all $i \neq j$, we have:

$$P_{ij}(t) = \left(\sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} \right) - \left(\sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k} \right)$$

$$P_{ii}(t) = e^{-\lambda_i t}$$

Example 6.8 The Yule process

We recall that a Yule process is a pure birth process $\{X(t) : t \geq 0\}$ with:

$$\lambda_n = \lambda n, \quad \text{for all } n \geq 0$$

We then compute the transition probability $P_{1j}(t)$, where $j > i$ using Proposition 6.1 with $i = 1$:

$$P_{1j}(t) = \left(\sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} \right) - \left(\sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k} \right)$$

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$$= \left(\sum_{k=1}^{j} e^{-k\lambda t} \prod_{r \neq k, r=1}^{j} \frac{r}{r - k} \right) - \left(\sum_{k=1}^{j-1} e^{-k\lambda t} \prod_{r \neq k, r=1}^{j-1} \frac{r}{r - k} \right)$$

Example 6.8 The Yule process (cont.)

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$$= e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\frac{j}{j-k} - 1 \right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

Example 6.8 The Yule process (cont.)

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$$= e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\prod_{r \neq k, r=1}^{j} \frac{r}{r-k} - \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k} \right)$$

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Example 6.8 The Yule process (cont.)

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$$= e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\prod_{r\neq k, r=1}^{j} \frac{r}{r-k} - \prod_{r\neq k, r=1}^{j-1} \frac{r}{r-k}\right)$$

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We take a closer look at the two red products in the above expression:

$$\prod_{r=1}^{j-1} \frac{r}{r-j}$$

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$$\prod_{r=1}^{j-1} \frac{r}{r-j} = (-1)^{j-1} \cdot \prod_{r=1}^{j-1} \frac{r}{j-r}$$

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$$= (-1)^{j-1} \cdot \frac{1 \cdot 2 \cdot \dots \cdot (j-1)}{(j-1)(j-2) \cdots 2 \cdot 1}$$

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$$= (-1)^{j-1} \cdot \frac{1 \cdot 2 \cdot \cdots \cdot (j-1)}{(j-1)(j-2) \cdots 2 \cdot 1}$$

$$= (-1)^{j-1}$$

$$= (-1)^{j-1} \binom{j-1}{j-1} \qquad \text{since } \binom{j-1}{j-1} = 1$$

Example 6.8 The Yule process (cont.)

$$\left(\frac{k}{j-k}\right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

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$$\left(\frac{k}{j-k}\right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

$$= \frac{k \cdot [1 \cdot 2 \cdot \cdots \cdot (k-1) \cdot (k+1) \cdots (j-1)]}{(j-k) \cdot [(1-k) \cdots ((k-1)-k) \cdot ((k+1)-k) \cdots ((j-1)-k)]}$$

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$$= (-1)^{k-1} \frac{(j-1)!}{(k-1)(k-2) \cdots 1 \cdot (j-k)!}$$

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$$\left(\frac{k}{j-k}\right)\prod_{r\neq k, r=1}^{j-1}\frac{r}{r-k}$$

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$$= (-1)^{k-1}\frac{(j-1)!}{(k-1)(k-2)\cdots 1\cdot(j-k)!} = (-1)^{k-1}\frac{(j-1)!}{(k-1)!\cdot(j-k)!}$$

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$$= (-1)^{k-1}\frac{(j-1)!}{(k-1)(k-2)\cdots 1\cdot(j-k)!} = (-1)^{k-1}\frac{(j-1)!}{(k-1)!\cdot(j-k)!}$$

$$= (-1)^{k-1}\binom{j-1}{k-1} \qquad k = 1, \ldots, (j-1).$$

Example 6.8 The Yule process (cont.)

$$P_{1j}(t) = e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\frac{k}{j-k} \right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

$$= e^{-j\lambda t} (-1)^{j-1} \binom{j-1}{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1}$$

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$$= \sum_{k=1}^{j} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1} = \sum_{i=0}^{j-1} e^{-(i+1)\lambda t} (-1)^{i} \binom{j-1}{i}$$

Example 6.8 The Yule process (cont.)

$$P_{1j}(t) = e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\frac{k}{j-k}\right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

$$= e^{-j\lambda t} (-1)^{j-1} \binom{j-1}{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1}$$

$$= \sum_{k=1}^{j} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1} = \sum_{i=0}^{j-1} e^{-(i+1)\lambda t} (-1)^{i} \binom{j-1}{i}$$

$$= e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} e^{-i\lambda t} (-1)^{i}$$

Example 6.8 The Yule process (cont.)

$$P_{1j}(t) = e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\frac{k}{j-k}\right) \prod_{r\neq k, r=1}^{j-1} \frac{r}{r-k}$$

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$$= \sum_{k=1}^{j} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1} = \sum_{i=0}^{j-1} e^{-(i+1)\lambda t} (-1)^{i} \binom{j-1}{i}$$

$$= e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} e^{-i\lambda t} (-1)^{i} = e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} (-e^{-\lambda t})^{i} \cdot 1^{(j-1)-i}$$

Example 6.8 The Yule process (cont.)

$$P_{1j}(t) = e^{-j\lambda t} \prod_{r=1}^{j-1} \frac{r}{r-j} + \sum_{k=1}^{j-1} e^{-k\lambda t} \left(\frac{k}{j-k}\right) \prod_{r \neq k, r=1}^{j-1} \frac{r}{r-k}$$

$$= e^{-j\lambda t} (-1)^{j-1} \binom{j-1}{j-1} + \sum_{k=1}^{j-1} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1}$$

$$= \sum_{k=1}^{j} e^{-k\lambda t} (-1)^{k-1} \binom{j-1}{k-1} = \sum_{i=0}^{j-1} e^{-(i+1)\lambda t} (-1)^{i} \binom{j-1}{i}$$

$$= e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} e^{-i\lambda t} (-1)^{i} = e^{-\lambda t} \sum_{i=0}^{j-1} \binom{j-1}{i} (-e^{-\lambda t})^{i} \cdot 1^{(j-1)-i}$$

$$= e^{-\lambda t} (1 - e^{-\lambda t})^{j-1} \qquad \text{(by the binomial formula)}$$

Example 6.8 The Yule process (cont.)

Thus, we have shown that:

$$P(X(t) = j | X(0) = 1) = P_{1j}(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}, \quad t > 0, \quad j = 1, 2, \ldots$$

Hence, we have that:

$$(X(t)|X(0) = 1) \sim Geom(e^{-\lambda t})$$

$$E[X(t)|X(0) = 1] = (e^{-\lambda t})^{-1} = e^{\lambda t}$$

Example 6.8 The Yule process (cont.)

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In the general case where $X(0) = i$, each individual in the initial population generates offsprings independently of the others.

Example 6.8 The Yule process (cont.)

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In the general case where $X(0) = i$, each individual in the initial population generates offsprings independently of the others.

Hence, in this case $(X(t) | X(0) = i)$ has the distribution of a sum of i independent and geometrically distributed variables, i.e., a negative binomial distribution.

Example 6.8 The Yule process (cont.)

The following result summarizes these results:

Proposition

Let $\{X(t) : t \geq 0\}$ *be a Yule process with rate* λ*. Then we have:*

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-i\lambda t}(1 - e^{-\lambda t})^{j-i}, \quad t > 0, \quad 1 \leq i \leq j$$

$$E[X(t)|X(0) = i] = i \cdot e^{\lambda t}, \quad t > 0, \quad i = 1, 2, \ldots$$