STK2130 – Chapter 6.8

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6.8 Uniformization

In this section we consider the special case where the Markov chain $\{X(t) : t \geq 0\}$, with state space $\mathcal{X}$, has the property that:

$$v_i = v, \quad \text{for all } i \in \mathcal{X},$$

where v_i as usual denotes the transition rate in state i, $i \in \mathcal{X}$.

We can the introduce a new process $\{N(t) : t \geq 0\}$, where:

$$N(t) = \text{The number of transitions in } [0, t], \quad t \geq 0.$$

It is then easy to see that $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate v.

6.8 Uniformization (cont.)

We then derive an expression for the transition probabilities by conditioning on $N(t)$:

$$P_{ij}(t) = P(X(t) = j | X(0) = i)$$

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$$= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n | X(0) = i)$$

6.8 Uniformization (cont.)

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$$
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6.8 Uniformization (cont.)

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&= \sum_{n=0}^{\infty} P(X(t) = j | X(0) = i, N(t) = n) \cdot P(N(t) = n) \\
&= \sum_{n=0}^{\infty} Q_{ij}^{n} \cdot \frac{(vt)^n}{n!} e^{-vt}
\end{aligned}$$

where Q_{ij}^n denotes the *n-step transition probability* from state i to state j for the built-in discete-time Markov chain.

6.8 Uniformization (cont.)

Since $P(N(t) = n)$ typically is small if n is large, we have the following approximation:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^{n} \cdot \frac{(vt)^n}{n!} e^{-vt}$$

provided that N is large.

6.8 Uniformization (cont.)

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NOTE: If the built-in discete-time Markov chain is ergodic, i.e., irreducible, positive recurrent and aperiodic, we have, we have:

$$\lim_{n \to \infty} Q_{ij}^{n} = \pi_j, \quad j \in \mathcal{X}.$$

Hence, the approximation can be improved by using:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^{n} \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot P(N(t) > N).$$

6.8 Uniformization (cont.)

In fact we have:

$$P_{ij}(t) \approx \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot P(N(t) > N)$$

$$= \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j \cdot [1 - P(N(t) \leq N)]$$

$$= \sum_{n=0}^{N} Q_{ij}^n \cdot \frac{(vt)^n}{n!} e^{-vt} + \pi_j - \pi_j \sum_{n=0}^{N} \frac{(vt)^n}{n!} e^{-vt}$$

$$= \pi_j + \sum_{n=0}^{N} (Q_{ij}^n - \pi_j) \frac{(vt)^n}{n!} e^{-vt}$$

which typically is a very good approximation even for moderately sized N.

6.8 Uniformization (cont.)

Assume (far) more generally that $v_i \leq v$ for all $i \in \mathcal{X}$, and let:

$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} Q_{ij} & j \neq i \end{cases}$$

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If the chain is in state i, the probability that a transition results in a state change is v_i/v, while the probability of no state change is $1 - v_i/v$.

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Given that a transition results in a state change from state i, the probability that the next state is state j is Q_{ij} as before.

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Given that a transition results in a state change from state i, the probability that the next state is state j is Q_{ij} as before.

The unconditional probability of a transition from state i to state j is then Q^*_{ij}.

6.8 Uniformization (cont.)

Replacing the Q_{ij}s by the Q^*_{ij}s in the formula for the transition probabilities, we get:

$$P_{ij}(t) = \sum_{n=0}^{\infty} Q^{*n}_{ij} \cdot \frac{(vt)^n}{n!} e^{-vt}$$

Note that if $v_i = v$ for all $i \in \mathcal{X}$, we get:

$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v}, & j = i \\ \frac{v_i}{v} Q_{ij}, & j \neq i \end{cases} = \begin{cases} 0, & j = i \\ Q_{ij}, & j \neq i \end{cases}$$

Example 6.23

The lifetimes and repair times of a system are independent and exponentially distributed with rates respectively $v_1 = \lambda$ and $v_0 = \mu$. (See Example 6.11.)

The system is modelled as a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1\}$, where[1]:

$$X(t) = I(\text{The system is functioning at time } t), \quad t \geq 0.$$

The matrix of transition probabilities of the built-in discrete-time Markov chain is:

$$\boldsymbol{Q} = \begin{bmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[1] In the Ross(2019) state 0 is the functioning state and state 1 is the failed state.

Example 6.23 (cont.)

A uniformized version of this model, is obtained by letting $v = \lambda + \mu$, and:

$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} Q_{ij} & j \neq i \end{cases}$$

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$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} Q_{ij} & j \neq i \end{cases}$$

Using that $v_0 = \mu$ and $v_1 = \lambda$, we get:

$$Q^*_{00} = 1 - \frac{v_0}{v} = 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

$$Q^*_{01} = \frac{v_0}{v} Q_{01} = \frac{\mu}{\lambda + \mu} \cdot 1 = \frac{\mu}{\lambda + \mu}$$

$$Q^*_{10} = \frac{v_1}{v} Q_{10} = \frac{\lambda}{\lambda + \mu} \cdot 1 = \frac{\lambda}{\lambda + \mu}$$

$$Q^*_{11} = 1 - \frac{v_1}{v} = 1 - \frac{\lambda}{\lambda + \mu} = \frac{\mu}{\lambda + \mu}$$

Example 6.23 (cont.)

In matrix form we get:

$$\boldsymbol{Q}^* = \begin{bmatrix} Q^*_{00} & Q^*_{01} \\ Q^*_{10} & Q^*_{11} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix} = \begin{bmatrix} a & (1-a) \\ a & (1-a) \end{bmatrix}$$

where we have introduced $a = \lambda/(\lambda+\mu)$.

From this it follows that the 2-step transition probability matrix is:

$$\boldsymbol{Q}^{*(2)} = \boldsymbol{Q}^* \cdot \boldsymbol{Q}^* = \begin{bmatrix} a & (1-a) \\ a & (1-a) \end{bmatrix} \cdot \begin{bmatrix} a & (1-a) \\ a & (1-a) \end{bmatrix}$$

$$= \begin{bmatrix} (a+(1-a))a & (a+(1-a))(1-a) \\ (a+(1-a))a & (a+(1-a))(1-a) \end{bmatrix} = \begin{bmatrix} a & (1-a) \\ a & (1-a) \end{bmatrix} = \boldsymbol{Q}^*$$

Repeating this argument, we get that $\boldsymbol{Q}^{*(n)} = \boldsymbol{Q}^*$, $n = 1, 2 \ldots$.

Example 6.23 (cont.)

We also recall that:

$$\boldsymbol{Q}^{*(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}$$

Using the formula for the transition probabilities, we get:

$$P_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{*n} \cdot \frac{(vt)^n}{n!} e^{-vt} = Q_{ij}^{*0} \cdot e^{-vt} + \sum_{n=1}^{\infty} Q_{ij}^{*n} \cdot \frac{(vt)^n}{n!} e^{-vt}$$

$$= Q_{ij}^{*0} \cdot e^{-(\lambda+\mu)t} + Q_{ij}^{*} \cdot \sum_{n=1}^{\infty} \frac{((\lambda+\mu)t)^n}{n!} e^{-(\lambda+\mu)t}$$

$$= I(i = j) \cdot e^{-(\lambda+\mu)t} + Q_{ij}^{*} \cdot (1 - e^{-(\lambda+\mu)t})$$

Example 6.23 (cont.)

We then use that:

$$\boldsymbol{Q}^* = \begin{bmatrix} Q^*_{00} & Q^*_{01} \\ Q^*_{10} & Q^*_{11} \end{bmatrix} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

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Inserting this we get:

$$P_{00}(t) \;=\; e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) \;=\; \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

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Example 6.23 (cont.)

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$$P_{10}(t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

$$P_{11}(t) = e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

Example 6.24

We consider the same two-state system as in Example 6.23, and assume that $X(0) = 1$. We then define:

$$U(t) = \int_0^t X(s)ds = \text{The fraction of the interval } [0, t] \text{ where } X(s) = 1$$

We can then calculate $E[U(t)]$ as follows:

$$\begin{aligned} E[U(t)] &= E\left[\int_0^t X(s)ds\right] = \int_0^t E[X(s)]ds \\ &= \int_0^t P(X(s) = 1 | X(0) = 1)ds = \int_0^t P_{11}(s)ds \end{aligned}$$

Example 6.24 (cont.)

Hence, since we have shown that:

$$P_{11}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

we get that:

$$E[U(t)] = \int_0^t \left[\frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)s} \right] ds$$

$$= \frac{\mu t}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2} [1 - e^{-(\lambda+\mu)t}]$$

Example 6.24 (cont.)

Hence, since we have shown that:

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we get that:

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$$= \frac{\mu t}{\lambda + \mu} + \frac{\lambda}{(\lambda + \mu)^2}[1 - e^{-(\lambda + \mu)t}]$$

We note that this also implies that:

$$\lim_{t \to \infty} E\left[\frac{U(t)}{t}\right] = \frac{\mu}{\lambda + \mu} = \lim_{t \to \infty} P_{11}(t)$$