STK2130 – Chapter 6.9

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6.9 Computing Transition Probabilities

We start out by introducing the following notation:

$$r_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases}$$

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Kolmogorov's backward equations can then be written as:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) + v_i P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X}} r_{ik} P_{kj}(t)$$

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Similarly, Kolmogorov's forward equations can then be written as:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} P_{ik}(t) q_{kj} + P_{ij}(t) v_j$$

$$= \sum_{k \in \mathcal{X}} P_{ik}(t) r_{kj}$$

6.9 Computing Transition Probabilities (cont.)

Now, let $\boldsymbol{R} = [r_{ij}]_{i,j \in \mathcal{X}}$ be the matrix of the r_{ij}'s.

Then Kolmogorov's backward equations can then be written in matrix form as:

$$\boldsymbol{P}'(t) = \boldsymbol{R}\boldsymbol{P}(t)$$

while Kolmogorov's forward equations can then be written in matrix form as:

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Both sets of equations can be viewed as a matrix version of a scalar differential equation of the form:

$$P'(t) = rP(t) = P(t)r$$

This scalar differential equation has the solution $P(t) = P(0)e^{rt}$.

6.9 Computing Transition Probabilities (cont.)

It can be shown that Kolmogorov's backward and forward equations have a similar solution:

$$\boldsymbol{P}(t) = \boldsymbol{P}(0)e^{\boldsymbol{R}t}$$

Using the boundary condition that $\boldsymbol{P}(0) = \boldsymbol{I}$, we get that:

$$\boldsymbol{P}(t) = e^{\boldsymbol{R}t},$$

where the matrix $e^{\boldsymbol{R}t}$ is given by:

$$e^{\boldsymbol{R}t} = \sum_{n=0}^{\infty} \boldsymbol{R}^n \frac{t^n}{n!} = \lim_{n\to\infty} \left(\boldsymbol{I} + \boldsymbol{R} \cdot \frac{t}{n}\right)^n \approx \left(\boldsymbol{I} + \boldsymbol{R} \cdot \frac{t}{N}\right)^N$$

where N is large.