STK2130 – Chapter 6 overview

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Discrete-time Markov Chains

We recall from Chapter 4:

Let $\{X_n : n \geq 0\}$ be a discrete-time stochastic process with discrete state space $\mathcal{X}$.

The process is a Markov chain if for $n = 1, 2, \ldots$ we have:

$$P(X_{n+1} = j | X_n = i, X_u = x_u, 0 \leq u < n)$$

$$= P(X_{n+1} = j | X_n = i), \quad i, j, x_u \in \mathcal{X}$$

If we also have that $P(X_{n+1} = j | X_n = i)$ is independent of n, then the Markov chain is said to have stationary (or homogeneous) transition probabilities.

6.2 Continuous-Time Markov Chains

Let $\{X(t) : t \geq 0\}$ be a continuous-time stochastic process with discrete state space $\mathcal{X}$.

The process is a Markov chain if for $s, t > 0$ we have:

$$P(X(t+s) = j | X(s) = i,\ X(u) = x(u),\ 0 \leq u < s)$$

$$= P(X(t+s) = j | X(s) = i), \quad i, j, x(u) \in \mathcal{X}$$

If we also have that $P(X(t+s) = j | X(s) = i)$ is independent of s, then the Markov chain is said to have stationary (or homogeneous) transition probabilities.

6.2 Continuous-Time Markov Chains (cont.)

ALTERNATIVE DEFINITION:
A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables (because of the Markov property).
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with rate v_i (because of the Markov property and stationarity).
- When the process leaves state i, it enters state j with some transition probability Q_{ij} where:

$$Q_{ii} = 0, \quad \text{for all } i \in \mathcal{X}$$

$$\sum_{j \in \mathcal{X}} Q_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.

6.3 Birth and Death Processes

A Birth and Death Process $\{X(t) : t \geq 0\}$ has state space $\mathcal{X} = \{0, 1, 2, \ldots\}$.

Assume that $X(t) = n > 0$. Then the next transition is determined as follows:

- Sample $V \sim exp(\lambda_n)$ and $W \sim exp(\mu_n)$ independent of each other with respective outcomes v and w.
- If $v < w$ then the process transits to state $n + 1$ at time $t + v$, i.e., $X(t + v) = n + 1$. This called a birth.
- If $w < v$ then the process transits to state $n - 1$ at time $t + w$, i.e., $X(t + w) = n - 1$. This called a death.

NOTE: When $X(t) = 0$, only births are possible, so in this case we assume that $W = \infty$, which corresponds to the rate μ_0 being zero, and $P_{01} = 1$.

6.3 Birth and Death Processes

We consider a general birth and death process, $\{X(t) : t \geq 0\}$, with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_0, \mu_1, \ldots$, where $\mu_0 = 0$.

Assume that $X(0) = i$, where $i \geq 0$, and define T_i to be the time until the process enters state $i + 1$ for the first time.

GOAL: Calculate $E[T_i]$.

Since $T_0 \sim exp(\lambda_0)$, we know that:

$$E[T_0] = \frac{1}{\lambda_0}.$$

6.3 Birth and Death Processes

By using a recursive relation, and that $E[T_0] = \lambda_0^{-1}$, we get:

$$E[T_0] = \frac{1}{\lambda_0}$$

$$E[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}$$

$$E[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}[\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}\frac{1}{\lambda_0}]$$

$$\ldots$$

6.4 The Transition Probability Function $P_{ij}(t)$

The transition probabilities of a stationary continuous-time Markov chain $\{X(t) : t \geq 0\}$, with state space $\mathcal{X}$ are defined as:

$$P_{ij}(t) = P(X(t+s) = j | X(s) = i), \quad t \geq 0, \quad i, j \in \mathcal{X}$$

Proposition (6.1)

For a pure birth process where $\lambda_i \neq \lambda_j$ for all $i \neq j$, we have:

$$P_{ij}(t) = \left(\sum_{k=i}^{j} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j} \frac{\lambda_r}{\lambda_r - \lambda_k} \right) - \left(\sum_{k=i}^{j-1} e^{-\lambda_k t} \prod_{r \neq k, r=i}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k} \right)$$

$$P_{ii}(t) = e^{-\lambda_i t}$$

The Yule Process

Assume that $\{X(t) : t \geq 0\}$ is a birth and death process with:

$$\mu_n = 0, \quad \text{for all } n \geq 0$$
$$\lambda_n = \lambda n, \quad \text{for all } n \geq 0$$

Since the death rate is zero, this is a pure birth process. The birth rate λn is proportional to the state, i.e., number of individuals in the population.

This implies that the time the process stays in state n is exponentially distributed with rate λn. Thus, the expected time between transitions becomes smaller and smaller as n grows.

The Yule process (cont.)

Proposition

Let $\{X(t) : t \geq 0\}$ *be a Yule process with rate* λ*. Then we have:*

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-i\lambda t}(1 - e^{-\lambda t})^{j-i}, \quad t > 0, \quad 1 \leq i \leq j$$

$$E[X(t)|X(0) = i] = i \cdot e^{\lambda t}, \quad t > 0, \quad i = 1, 2, \ldots$$

6.4 Kolmogorov's Backward Equations

A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables (because of the Markov property).
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with rate v_i (because of the Markov property and stationarity).
- When the process leaves state i, it enters state j with some transition probability Q_{ij} where:

$$Q_{ii} = 0, \quad \text{for all } i \in \mathcal{X}$$

$$\sum_{j \in \mathcal{X}} Q_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.

6.4 Kolmogorov's Backward Equations

We now introduce the following notation:

$$q_{ij} = v_i Q_{ij}, \quad i, j \in \mathcal{X}.$$

INTERPRETATION: Since v_i is the rate at which the process makes a transition when in state i and Q_{ij} is the probability that this transition is into state j, it follows that q_{ij} is the rate, when in state i, at which the process makes a transition into state j.

The quantities q_{ij} are called the instantaneous transition rates.

Since we have:

$$v_i = v_i \sum_{j \in \mathcal{X}} Q_{ij} = \sum_{j \in \mathcal{X}} v_i Q_{ij} = \sum_{j \in \mathcal{X}} q_{ij},$$

$$Q_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_{j \in \mathcal{X}} q_{ij}},$$

the probabilistic properties of $\{X(t) : t \geq 0\}$ is determined by the q_{ij}'s.

6.4 Kolmogorov's Backward Equations

Theorem (6.1 – Kolmogorov's backward equations)

For all $t \geq 0$ and states $i, j \in \mathcal{X}$ we have:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

PROOF: By Lemma 6.3 we have:

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t) - P_{ij}(t) \\ &= \sum_{k \in \mathcal{X} \setminus i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t) - P_{ij}(t) \\ &= \sum_{k \in \mathcal{X} \setminus i} P_{ik}(h) P_{kj}(t) - [1 - P_{ii}(h)] P_{ij}(t) \end{aligned}$$

By dividing both sides by h and letting $h \to 0$, we can use Lemma 6.2 and get:

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \in \mathcal{X} \setminus i} \lim_{h \to 0} \frac{P_{ik}(h)}{h} P_{kj}(t) - \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

■

6.4 Kolmogorov's Forward Equations

Theorem (6.2 – Kolmogorov's forward equations)

For all $t \geq 0$ and states $i, j \in \mathcal{X}$ we have:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

PROOF: By Lemma 6.3 we have:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) P_{kj}(h) + P_{ij}(t) P_{jj}(h) - P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) P_{kj}(h) - P_{ij}(t)[1 - P_{jj}(h)]$$

6.4 Kolmogorov's Forward Equations (cont.)

By dividing both sides by h and letting $h \to 0$, we can use Lemma 6.2 and get:

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) \lim_{h \to 0} \frac{P_{kj}(h)}{h} - P_{ij}(t) \lim_{h \to 0} \frac{1 - P_{jj}(h)}{h}$$

$$= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$$

■

6.4 Kolmogorov's Equations

We again assume that $\mathcal{X} = \{1, 2, \ldots, n\}$, and recall the following matrices:

$$\boldsymbol{R} = \begin{bmatrix} -v_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ q_{2,1} & -v_2 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & -v_3 & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & -v_n \end{bmatrix}$$

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{1,1}(t) & P_{1,2}(t) & P_{1,1}(t) & \cdots & P_{1,n}(t) \\ P_{2,1}(t) & P_{2,2}(t) & P_{2,3}(t) & \cdots & P_{2,n}(t) \\ P_{3,1}(t) & P_{3,2}(t) & P_{3,3}(t) & \cdots & P_{3,n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n,1}(t) & P_{n,2}(t) & P_{n,3}(t) & \cdots & P_{n,n}(t) \end{bmatrix}$$

6.4 Kolmogorov's Equations

Kolmogorov's backward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

can be written in the following form:

$$\boldsymbol{P}'(t) = \boldsymbol{R}\boldsymbol{P}(t).$$

Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

can be written in the following form:

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R}.$$

Remaining lectures - Chapter 6

- Week 18
  - Chapter 6.5 Limiting Probabilities
  - Chapter 6.8 Uniformization
  - Chapter 6.9 Computing the Transition Probabilities

Remaining lectures - Chapter 7

- Week 19
  - Chapter 7.1 Renewal Theory and Its Applications
  - Chapter 7.2 Distribution of $N(t)$

Remaining lectures - Chapter 10

- Week 20
  - Chapter 10.1 Brownian Motion
  - Chapter 10.2 Hitting Times, Maximum Variable, and the Gambler's Ruin Problem
  - Chapter 10.3 Variations on Brownian Motion

EXAM: (Week 22) May 27, 14:30 – June 3, 14:30