STK2130 – Chapter 6 overview 2

A. B. Huseby

Department of Mathematics
University of Oslo, Norway

Chapter 6.5 Limiting Probabilities

To determine the limiting distribution, we use Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

By taking the limit on both sides when t goes to infinity, we get:

$$0 = \lim_{t \to \infty} P'_{ij}(t) = \lim_{t \to \infty} \left[\sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j \right]$$

$$= \sum_{k \in \mathcal{X} \setminus j} \pi_k q_{kj} - \pi_j v_j, \quad j \in \mathcal{X}.$$

Combined with the equation $\sum_{j \in \mathcal{X}} \pi_j = 1$, we can determine the limiting distribution.

6.5 Limiting Probabilities (cont.)

In the case where $\mathcal{X} = \{1, \ldots, n\}$ we introduce:

$$\boldsymbol{R} = \begin{bmatrix} -v_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ q_{2,1} & -v_2 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & -v_3 & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & -v_n \end{bmatrix}$$

and let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$. Then the equations:

$$\sum_{k \in \mathcal{X} \setminus j} \pi_k q_{kj} - \pi_j v_j = 0, \quad j \in \mathcal{X}.$$

can be written as:

$$\boldsymbol{\pi}\boldsymbol{R} = \mathbf{0}.$$

where $\mathbf{0} = (0, \ldots, 0)$.

6.5 Limiting Probabilities (cont.)

The limiting distribution for continuous-time Markov chains is found by using the following equations:

$$\boldsymbol{\pi} \boldsymbol{R} = \mathbf{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

6.5 Limiting Probabilities (cont.)

The limiting distribution for continuous-time Markov chains is found by using the following equations:

$$\boldsymbol{\pi} \boldsymbol{R} = \boldsymbol{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

We compare this to the equations we use for discrete-time Markov chains:

$$\boldsymbol{\pi} \boldsymbol{P} = \boldsymbol{\pi}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

or equivalently:

$$\boldsymbol{\pi}(\boldsymbol{P} - \boldsymbol{I}) = \boldsymbol{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

where $\boldsymbol{P}$ denotes the matrix of transition probabilities for the chain.

6.5 Limiting Probabilities (cont.)

When the limiting probabilities exist, we say that the chain is ergodic.

Necessary and sufficient conditions for the existence of the limiting distribution are:

- All states of the Markov chain communicate in the sense that starting in state i there is a positive probability of ever being in state j, for all $i, j \in \mathcal{X}$.
- The Markov chain is positive recurrent in the sense that, starting in any state, the mean time to return to that state is finite.

If these conditions hold, the limiting probabilities exist and satisfy the derived equations.

In addition, the probability π_j also has the interpretation of being the long-run proportion of time that the process is in state j.

6.8 Uniformization

In this section we consider the special case where the Markov chain $\{X(t) : t \geq 0\}$, with state space $\mathcal{X}$, has the property that:

$$v_i = v, \quad \text{for all } i \in \mathcal{X},$$

where v_i as usual denotes the transition rate in state i, $i \in \mathcal{X}$.

We can the introduce a new process $\{N(t) : t \geq 0\}$, where:

$$N(t) = \text{The number of transitions in } [0, t], \quad t \geq 0.$$

It is then easy to see that $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate v.

6.8 Uniformization (cont.)

We then derive an expression for the transition probabilities by conditioning on $N(t)$:

$$
\begin{aligned}
P_{ij}(t) &= P(X(t) = j|X(0) = i) \\
&= \sum_{n=0}^{\infty} P(X(t) = j|X(0) = i, N(t) = n) \cdot P(N(t) = n|X(0) = i) \\
&= \sum_{n=0}^{\infty} P(X(t) = j|X(0) = i, N(t) = n) \cdot P(N(t) = n) \\
&= \sum_{n=0}^{\infty} Q_{ij}^{n} \cdot \frac{(vt)^n}{n!} e^{-vt}
\end{aligned}
$$

where Q_{ij}^{n} denotes the n-step transition probability from state i to state j for the built-in discete-time Markov chain.

6.8 Uniformization

Assume (far) more generally that $v_i \leq v$ for all $i \in \mathcal{X}$, and let:

$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v} & j = i \\ \frac{v_i}{v} Q_{ij} & j \neq i \end{cases}$$

$\{X(t) : t \geq 0\}$ can now be interpreted as a Markov chain, where the transition rate is v for all states $i \in \mathcal{X}$. However, only a fraction of the transitions results in actual state changes.

If the chain is in state i, the probability that a transition results in a state change is v_i/v, while the probability of no state change is $1 - v_i/v$.

Given that a transition results in a state change from state i, the probability that the next state is state j is Q_{ij} as before.

The unconditional probability of a transition from state i to state j is then Q^*_{ij}.

6.8 Uniformization

Replacing the Q_{ij}s by the Q^*_{ij}s in the formula for the transition probabilities, we get:

$$P_{ij}(t) = \sum_{n=0}^{\infty} Q^{*n}_{ij} \cdot \frac{(vt)^n}{n!} e^{-vt}$$

Note that if $v_i = v$ for all $i \in \mathcal{X}$, we get:

$$Q^*_{ij} = \begin{cases} 1 - \frac{v_i}{v}, & j = i \\ \frac{v_i}{v} Q_{ij}, & j \neq i \end{cases} = \begin{cases} 0, & j = i \\ Q_{ij}, & j \neq i \end{cases}$$

6.9 Computing the Transition Probabilities

We start out by introducing the following notation:

$$r_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases}$$

Kolmogorov's backward equations can then be written as:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) + v_i P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X}} r_{ik} P_{kj}(t)$$

Similarly, Kolmogorov's forward equations can then be written as:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} P_{ik}(t) q_{kj} + P_{ij}(t) v_j$$

$$= \sum_{k \in \mathcal{X}} P_{ik}(t) r_{kj}$$

6.9 Computing the Transition Probabilities

Now, let $\boldsymbol{R} = [r_{ij}]_{i,j \in \mathcal{X}}$ be the matrix of the r_{ij}'s.

Then Kolmogorov's backward equations can then be written in matrix form as:

$$\boldsymbol{P}'(t) = \boldsymbol{R}\boldsymbol{P}(t)$$

while Kolmogorov's forward equations can then be written in matrix form as:

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R}$$

Both sets of equations can be viewed as a matrix version of a scalar differential equation of the form:

$$P'(t) = rP(t) = P(t)r$$

This scalar differential equation has the solution $P(t) = P(0)e^{rt}$.

6.9 Computing Transition Probabilities (cont.)

It can be shown that Kolmogorov's backward and forward equations have a similar solution:

$$\boldsymbol{P}(t) = \boldsymbol{P}(0)e^{\boldsymbol{R}t}$$

Using the boundary condition that $\boldsymbol{P}(0) = \boldsymbol{I}$, we get that:

$$\boldsymbol{P}(t) = e^{\boldsymbol{R}t},$$

where the matrix $e^{\boldsymbol{R}t}$ is given by:

$$e^{\boldsymbol{R}t} = \sum_{n=0}^{\infty} \boldsymbol{R}^n \frac{t^n}{n!} = \lim_{n\to\infty} \left(\boldsymbol{I} + \boldsymbol{R} \cdot \frac{t}{n}\right)^n \approx \left(\boldsymbol{I} + \boldsymbol{R} \cdot \frac{t}{N}\right)^N$$

where N is large.

Remaining lectures - Chapter 7

- Week 19
  - Chapter 7.1 Renewal Theory and Its Applications
  - Chapter 7.2 Distribution of $N(t)$

Remaining lectures - Chapter 10

- Week 20
  - Chapter 10.1 Brownian Motion
  - Chapter 10.2 Hitting Times, Maximum Variable, and the Gambler's Ruin Problem
  - Chapter 10.3 Variations on Brownian Motion

EXAM: (Week 22) May 27, 14:30 – June 4, 14:30