STK2130 – Chapter 7.2

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7.2 Distribution of $N(t)$

In order to determine the distribution of $N(t)$, we note that:

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

Hence, we get:

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n + 1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where F_n denotes the distribution of S_n, i.e., the n-fold convolution of the distribution F.

Example 7.1

Assume that the interarrival distribution, F is geometric. That is:

$$P(X_n = i) = p(1-p)^{i-1}, \quad i = 1, 2, \ldots$$

Since sums of geometrically distributed variables have negative binomial distributions, we get:

$$P(S_n = k) = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & k \geq n \\ 0 & k < n \end{cases}$$

Example 7.1 (cont.)

From this we get that:

$$P(N(t) = n) = F_n(t) - F_{n+1}(t) = P(S_n \leq \lfloor t \rfloor) - P(S_{n+1} \leq \lfloor t \rfloor)$$

$$= \sum_{k=n}^{\lfloor t \rfloor} \binom{k-1}{n-1} p^n (1-p)^{k-n}$$

$$- \sum_{k=n+1}^{\lfloor t \rfloor} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

where $\lfloor t \rfloor = \max\{n \in \mathbb{N} : n \leq t\}$.

Example 7.1 (cont.)

In this case, however, we can find the distribution of $N(t)$ much easier by interpreting the process as an infinite series of binomial trials where p is the probability that an event occurs at a given point in time.

At time $t > 0$, the number of trials is $\lfloor t \rfloor$. Thus, $N(t) \sim Bin(\lfloor t \rfloor, p)$, and we have:

$$P(N(t) = n) = \binom{\lfloor t \rfloor}{n} p^n (1-p)^{\lfloor t \rfloor - n}, \quad n = 0, 1, \ldots, \lfloor t \rfloor.$$

7.2 Distribution of $N(t)$ (cont.)

We can also calculate $P(N(t) = n)$ by conditioning on S_n and get:

$$
\begin{aligned}
P(N(t) = n) &= \int_0^\infty P(N(t) = n | S_n = s) f_{S_n}(s) ds \\
&= \int_0^t P(N(t) = n | S_n = s) f_{S_n}(s) ds + \int_t^\infty 0 \cdot f_{S_n}(s) ds \\
&= \int_0^t P(X_{n+1} > t - s | S_n = s) f_{S_n}(s) ds \\
&= \int_0^t \bar{F}(t - s) f_{S_n}(s) ds
\end{aligned}
$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$.

Example 7.2

Assume that $X_n \sim exp(\lambda)$, $n = 1, 2, \ldots$. Then $S_n \sim Gamma(n, \lambda)$ $n = 1, 2, \ldots$.

By conditioning on S_n we get:

$$P(N(t) = n) = \int_0^t e^{-\lambda(t-s)} \cdot \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} ds$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \int_0^t s^{n-1} ds$$

$$= \frac{\lambda^n e^{-\lambda t}}{(n-1)!} \left[\frac{1}{n} s^n \right]_0^t$$

$$= \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Thus, we get the well-known result that $N(t) \sim Po(\lambda t)$.

The mean value of $N(t)$

We recall that:

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

By using this we can calculate $m(t) = E[N(t)]$ as:

$$m(t) = \sum_{k=1}^{\infty} k \cdot P(N(t) = k) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} P(N(t) = k)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(N(t) = k) = \sum_{n=1}^{\infty} P(N(t) \geq n)$$

$$= \sum_{n=1}^{\infty} P(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t)$$

The function $m(t)$ is called the renewal function of the process $\{N(t) : t \geq 0\}$.

Properties of $m(t)$

Proposition (The renewal function)

Let $m(t)$ be the renewal function of the renewal process $\{N(t) : t \geq 0\}$. Then the following holds:

- $m(t) < \infty, \quad$ *for all* $t < \infty$
- *The stochastic properties of* $\{N(t) : t \geq 0\}$ *are uniquely determined by* $m(t)$.

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NOTE: We have shown earlier that $P(N(t) < \infty) = 1$. From this result alone we cannot infer that $m(t) < \infty$ as well, as there are many distributions for which the mean values are infinite. Thus, the result that we in fact have $m(t) < \infty$ is a stronger result.

Properties of $m(t)$ (cont.)

EXAMPLE: Let $\{N(t) : t \geq 0\}$ be a homogeneous Poisson process with rate λ. Then we know that:

$$N(t) \sim Po(\lambda t)$$

Hence, it follows that:

$$m(t) = E[N(t)] = \lambda t$$

Since $m(t)$ uniquely determines the stochastic properties of $\{N(t) : t \geq 0\}$, it follows that no other renewal process can have a linear renewal function.

Integral equation for $m(t)$

We F denote the cumulative distribution, and f the density of X_1.

An integral equation for $m(t)$ can be found by conditioning on X_1:

$$\begin{aligned} m(t) &= \int_0^\infty E[N(t)|X_1 = x]f(x)dx \\ &= \int_0^t E[N(t)|X_1 = x]f(x)dx + \int_t^\infty 0 \cdot f(x)dx \\ &= \int_0^t [1 + E[N(t-x)]]f(x)dx \\ &= F(t) + \int_0^t m(t-x)f(x)dx \end{aligned}$$

This equation is called the renewal equation.

Example 7.3

Assume that $X_n \sim R[0,1]$ (the uniform distribution), and let $t \leq 1$. Then the renewal equation becomes:

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx$$

$$= t + \int_0^t m(t-x)dx = t + \int_0^t m(y)dy, \quad \text{by subst. } y = t - x.$$

By differentiating on both sides of this equation we get:

$$m'(t) = 1 + m(t)$$

By letting $h(t) = 1 + m(t)$, the equation becomes:

$$h'(t) = h(t)$$

Example 7.3 (cont.)

This is a homogeneous differential equation with solution $h(t) = Ce^t$, and hence

$$m(t) \;=\; Ce^t - 1$$

Since obviously $m(0) = C - 1 = 0$, the constant C must be 1. Hence, the renewal function becomes:

$$m(t) \;=\; e^t - 1, \quad 0 \leq t \leq 1.$$