STK2130 – Chapter 7 and 10 overview

A. B. Huseby

Department of Mathematics
University of Oslo, Norway

7.1 Renewal theory – Introduction

Definition

Let $\{N(t) : t \geq 0\}$ *be a counting process and let* X_n *denote the nth interarrival time, i.e., the time between the* $(n-1)$*st and the nth event of this process,* $n = 1, 2, \ldots$.

If $X_1, X_2, \ldots$ *are independent and identically distributed, then* $\{N(t) : t \geq 0\}$ *is said to be a renewal process.*

EXAMPLE: Consider a situation where we have an infinite supply of lightbulbs, and let X_n denote the lifetime of the nth lightbulb.

We use one lightbulb at a time. As soon as a lightbulb fails, it is immediately replaced by a new one.

We then introduce the counting process $\{N(t) : t \geq 0\}$ where $N(t)$ represents the number of failed lightbulbs at time t.

If $X_1, X_2, \ldots$ are independent and identically distributed, then $\{N(t) : t \geq 0\}$ is a renewal process.

7.1 Renewal theory – Introduction (cont.)

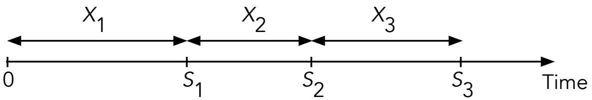

Figure: Renewal and interarrival times

For a renewal process the events are referred to as renewals.

If $\{N(t) : t \geq 0\}$ is a renewal process with interarrival times $X_1, X_2, \ldots$ we let:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots$$

That is, $S_1 = X_1$ is the time of the first renewal, $S_2 = X_1 + X_2$ is the time of the second renewal. In general S_n is the time of the nth renewal.

7.1 Renewal theory – Introduction (cont.)

We denote the cumulative distribution function of the interarrival times by F, and assume that:

$$F(0) = P(X_n = 0) < 1, \quad \text{and} \quad \lim_{t\to\infty} F(t) = P(X_n < \infty) = 1.$$

We also assume that $E[X_n] = \mu > 0$.

Under these assumptions we can show:

- $N(t) < \infty$ for all t with probability 1.
- $N(\infty) = \lim_{t\to\infty} N(t) = \infty$ with probability 1.

7.2 Distribution of $N(t)$

In order to determine the distribution of $N(t)$, we note that:

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

Hence, we get:

$$\begin{aligned} P(N(t) = n) &= P(N(t) \geq n) - P(N(t) \geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

where F_n denotes the distribution of S_n, i.e., the n-fold convolution of the distribution F.

7.2 Distribution of $N(t)$ (cont.)

We can also calculate $P(N(t) = n)$ by conditioning on S_n and get:

$$
\begin{aligned}
P(N(t) = n) &= \int_0^{\infty} P(N(t) = n | S_n = s) f_{S_n}(s) ds \\
&= \int_0^{t} P(N(t) = n | S_n = s) f_{S_n}(s) ds + \int_t^{\infty} 0 \cdot f_{S_n}(s) ds \\
&= \int_0^{t} P(X_{n+1} > t - s | S_n = s) f_{S_n}(s) ds \\
&= \int_0^{t} \bar{F}(t - s) f_{S_n}(s) ds
\end{aligned}
$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$.

The renewal function $m(t)$

Proposition (The renewal function)

Let $\{N(t) : t \geq 0\}$ be a renewal process, and let $m(t) = E[N(t)]$ be the renewal function. Then the following holds:

- $m(t) < \infty, \quad$ *for all* $t < \infty$
- *The stochastic properties of $\{N(t) : t \geq 0\}$ are uniquely determined by $m(t)$.*

NOTE: We have shown earlier that $P(N(t) < \infty) = 1$. From this result alone we cannot infer that $m(t) < \infty$ as well, as there are many distributions for which the mean values are infinite. Thus, the result that we in fact have $m(t) < \infty$ is a stronger result.

Integral equation for $m(t)$

We F denote the cumulative distribution, and f the density of X_1.

An integral equation for $m(t)$ can be found by conditioning on X_1:

$$\begin{aligned} m(t) &= \int_0^\infty E[N(t)|X_1 = x]f(x)dx \\ &= \int_0^t E[N(t)|X_1 = x]f(x)dx + \int_t^\infty 0 \cdot f(x)dx \\ &= \int_0^t [1 + E[N(t-x)]]f(x)dx \\ &= F(t) + \int_0^t m(t-x)f(x)dx \end{aligned}$$

This equation is called the renewal equation.

10.1 Brownian Motion

Definition (Brownian motion)

A Brownian motion is a stochastic process $\{X(t) : t \geq 0\}$ *where:*

(i) $X(0) = 0$

(ii) $\{X(t) : t \geq 0\}$ *has stationary and independent increments*

(iii) $X(t) \sim N(0, \sigma^2 t), \quad t > 0$

If $\sigma = 1$*,* $\{X(t) : t \geq 0\}$ *is called a standard Brownian motion.*

NOTE: If $\{Y(t) : t \geq 0\}$ is a Brownian motion, where $Y(t) \sim N(0, \sigma^2 t)$, then $\{X(t) : t \geq 0\}$, where $X(t) = Y(t)/\sigma$, for all $t \geq 0$ is a standard Brownian motion.

10.1 Brownian Motion (cont.)

We consider a standard Brownian motion $\{X(t) : t \geq 0\}$.

Let $0 < t_1 < t_2 < \cdots < t_n$, and let $X_i = X(t_i)$, $i = 1, 2, \ldots, n$

The joint density of $X_1, \ldots, X_n$ has the form:

$$f_{\boldsymbol{t}}(x_1, \ldots, x_n) = C(\boldsymbol{t})e^{-(1/2)Q(x_1,\ldots,x_n)}$$

where $C(\boldsymbol{t})$ is a suitable normalizing constant, and where:

$$Q(x_1, \ldots, x_n) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}$$

This formula is valid for any $n \geq 1$ and for any $0 < t_1 < \cdots < t_n$. Moreover, from this formula we can derive all possible conditional densities as well.

10.1 Brownian Motion (cont.)

EXAMPLE: Let $0 < t_1 < t_2$, and let $X_1 = X(t_1)$ and $X_2 = X(t_2)$. Then the joint density of X_1 and X_2 is:

$$f_{\boldsymbol{t}}(x_1, x_2) = C(\boldsymbol{t})e^{-(1/2)Q(x_1,x_2)}$$

where:

$$Q(x_1, x_2) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1}$$

The marginal densities of X_1 and X_2 are respectively:

$$f_{t_1}(x_1) = C(t_1)e^{-(1/2)(x_1^2/t_1)}$$

$$f_{t_2}(x_2) = C(t_2)e^{-(1/2)(x_2^2/t_2)}$$

The conditional density of X_2 given $X_1 = x_1$ then becomes:

$$f_{X_2|X_1=x_1} = \frac{f_{\boldsymbol{t}}(x_1, x_2)}{f_{t_1}(x_1)} = \frac{C(\boldsymbol{t})e^{-(1/2)\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1}\right]}}{C(t_1)e^{-(1/2)\left[\frac{x_1^2}{t_1}\right]}}$$

$$= C(t_2|t_1)e^{-(1/2)\left[\frac{(x_2-x_1)^2}{t_2-t_1}\right]}$$

where the normalizing constant $C(t_2|t_1) = C(\boldsymbol{t})/C(t_1)$.

By this and similar arguments we show that:

- $(X_2|X_1 = x_1) \sim N(x_1, t_2 - t_1)$.
- $(X_1|X_2 = x_2) \sim N(\frac{t_1}{t_2}x_2, \ \frac{t_1}{t_2}(t_2 - t_1))$.

10.2 Hitting Times, Max Variable and Ruin

Let $\{X(t) : t \geq 0\}$ be a Brownian motion process with variance parameter σ^2, and let:

$$T_a = \inf\{t > 0 : X(t) = a\} = \text{The first time the process hits } a.$$

We want to compute $P(T_a \leq t)$, where $a > 0$. In order to so, we instead consider $P(X(t) \geq a)$, and condition on the event $\{T_a \leq t\}$:

$$P(X(t) \geq a) = P(X(t) \geq a | T_a \leq t) P(T_a \leq t) + P(X(t) \geq a | T_a > t) P(T_a > t)$$

By symmetry, it follows that:

$$P(X(t) \geq a | T_a \leq t) = \frac{1}{2}$$

Moreover, we obviously have:

$$P(X(t) \geq a | T_a > t) = 0$$

Hitting Times, Max Variable and Ruin (cont.)

Hence, we have:

$$P(X(t) \geq a) = \tfrac{1}{2} P(T_a \leq t)$$

and thus:

$$P(T_a \leq t) = 2 \cdot P(X(t) \geq a) = 2 \cdot P(\frac{X(t)}{\sigma\sqrt{t}} \geq \frac{a}{\sigma\sqrt{t}}) = 2 \cdot \Phi(-\frac{a}{\sigma\sqrt{t}})$$

If $a < 0$, we can use a similar argument, and obtain:

$$P(T_a \leq t) = 2 \cdot P(X(t) \leq a) = 2 \cdot P(\frac{X(t)}{\sigma\sqrt{t}} \leq \frac{a}{\sigma\sqrt{t}}) = 2 \cdot \Phi(\frac{a}{\sigma\sqrt{t}})$$

The formulas can be combined to:

$$P(T_a \leq t) = 2 \cdot \Phi(-\frac{|a|}{\sigma\sqrt{t}})$$

Hitting Times, Max Variable and Ruin (cont.)

NOTE:

$$\text{If } a > 0: \qquad \max_{0 \leq s \leq t} X(s) \geq a \quad \Leftrightarrow \quad T_a \leq t$$

$$\text{If } a < 0: \qquad \min_{0 \leq s \leq t} X(s) \leq a \quad \Leftrightarrow \quad T_a \leq t$$

Hence, if $a > 0$, we have:

$$P(\max_{0 \leq s \leq t} X(s) \geq a) = P(T_a \leq t) = 2 \cdot \Phi(-\frac{a}{\sigma\sqrt{t}})$$

Similarly, if $a < 0$, we have:

$$P(\min_{0 \leq s \leq t} X(s) \leq a) = P(T_a \leq t) = 2 \cdot \Phi(\frac{a}{\sigma\sqrt{t}})$$

Hitting Times, Max Variable and Ruin (cont.)

Finally, we let $b < 0 < a$, and let $T = \min\{T_a, T_b\}$ where:

$$T_a = \inf\{t > 0 : X(t) = a\} = \text{The first time the process hits } a$$

$$T_b = \inf\{t > 0 : X(t) = b\} = \text{The first time the process hits } b$$

We want to calculate $P(T_a < T_b) = P(X(T) = a)$.

Let $P(X(T) = a) = p$, and $P(X(T) = b) = 1 - P(X(T) = a) = 1 - p$.

Since $E[X(t)] = 0$ for all $t \geq 0$, it follows that we in particular must have:

$$0 = E[X(T)] = a \cdot p + b \cdot (1 - p) = (a - b)p + b$$

By solving this equation with respect to p, we get:

$$P(T_a < T_b) = P(X(T) = a) = p = \frac{-b}{a - b} = \frac{|b|}{a + |b|}$$

10.3 Variations on Brownian Motion

- Brownian Motion with drift
- Geometric Brownian Motion

10.3.1 Brownian Motion with drift

We say that $\{X(t) : t \geq 0\}$ is a Brownian motion process with drift coefficient μ and variance parameter σ^2 if:

- $X(0) = 0$
- $\{X(t) : t \geq 0\}$ has stationary and independent increments
- $X(t) \sim N(\mu t, \sigma^2 t), \quad t \geq 0.$

An equivalent definition is to let $\{B(t) : t \geq 0\}$ be a standard Brownian motion process, and then define:

$$X(t) = \sigma B(t) + \mu t$$

10.3.2 Geometric Brownian Motion

If $\{Y(t) : t \geq 0\}$ is a Brownian motion process drift coefficient μ and variance parameter σ^2, then the process $\{X(t) : t \geq 0\}$ defined by:

$$X(t) = e^{Y(t)}$$

is called a geometric Brownian motion process.

If $\{X(t) : t \geq 0\}$ is geometric Brownian motion process, we obtain:

$$E[X(t)|X(u), 0 \leq u \leq s] = X(s)e^{\mu(t-s)+\sigma^2(t-s)/2}$$