STK2130 – Week 3

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Markov Chains

A discrete time, discrete space stochastic process $\{X_0, X_1, X_2, \ldots\}$ is called a (time homogenuous) *Markov chain* if:

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = P_{ij},$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$.

One-step transition matrix:

$$\boldsymbol{P} = \|P_{ij}\|_{i,j}$$

$P_{ij} \geq 0$ for all i, j and $\sum_j P_{ij} = 1$

Chapman-Kolmogorov Equations

***n*-step transition probabilities:**

$$P_{ij}^{n} = P\{X_{n+k} = j | X_k = i\}, \quad n \geq 1$$

***n*-step transition matrix:**

$$\boldsymbol{P}^{(n)} = \| P_{ij}^{n} \|_{i,j}$$

Chapman-Kolmogorov Equations:

$$P_{ij}^{n+m} = \sum_{k} P_{ik}^{n} \cdot P_{kj}^{m}$$

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^{(n)} \cdot \boldsymbol{P}^{(m)}$$

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^{(n-1)} \cdot \boldsymbol{P}^{(1)}$$

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^{n}$$

Example 4.8

$$P\{\text{Rain tomorrow}|\text{Rain today}\} = 0.75$$
$$P\{\text{Rain tomorrow}|\text{No rain today}\} = 0.35$$

$$\boldsymbol{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{bmatrix}$$

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{bmatrix}$$

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.5836 & 0.4164 \\ 0.5830 & 0.4170 \end{bmatrix}$$

Example 4.9

$$
\begin{aligned}
RR &= \text{Rain yesterday, Rain today,} \\
\bar{R}R &= \text{No rain yesterday, Rain today,} \\
R\bar{R} &= \text{Rain yesterday, No rain today,} \\
\bar{R}\bar{R} &= \text{No rain yesterday, No rain today.}
\end{aligned}
$$

$$
\begin{aligned}
P\{RR|RR\} &= 0.70 & \quad P\{R\bar{R}|RR\} &= 0.30 \\
P\{RR|\bar{R}R\} &= 0.50 & \quad P\{R\bar{R}|\bar{R}R\} &= 0.50 \\
P\{\bar{R}R|R\bar{R}\} &= 0.40 & \quad P\{\bar{R}\bar{R}|R\bar{R}\} &= 0.60 \\
P\{\bar{R}R|\bar{R}\bar{R}\} &= 0.20 & \quad P\{\bar{R}\bar{R}|\bar{R}\bar{R}\} &= 0.80
\end{aligned}
$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $R\bar{R}$, $\bar{R}\bar{R}$.

$$\boldsymbol{P} = \begin{bmatrix} 0.70 & 0.00 & 0.30 & 0.00 \\ 0.50 & 0.00 & 0.50 & 0.00 \\ 0.00 & 0.40 & 0.00 & 0.60 \\ 0.00 & 0.20 & 0.00 & 0.80 \end{bmatrix}$$

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} \color{red}{0.49} & \color{red}{0.12} & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix}$$

$$P\{\text{Rain Thursday}|\text{Rain Monday \& Rain Tuesday}\} = 0.49 + 0.12 = 0.61$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $\bar{R}R$, $\bar{R}\bar{R}$.

$$\boldsymbol{P}^{(7)} = \begin{bmatrix} 0.2723 & 0.1465 & 0.1580 & 0.4233 \\ 0.2633 & 0.1477 & 0.1549 & 0.4340 \\ 0.2441 & 0.1511 & 0.1477 & 0.4571 \\ 0.2352 & 0.1524 & 0.1447 & 0.4678 \end{bmatrix}$$

$$P\{\text{Rain next Tuesday}|\text{Rain Monday \& Rain Tuesday}\}$$

$$= 0.2723 + 0.1465 = 0.4188$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $\bar{R}R$, $\bar{R}\bar{R}$.

$$\boldsymbol{P}^{(20)} \approx \begin{bmatrix} 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \end{bmatrix}$$

$P\{\text{Rain "some day" in the future}\}$

$\approx P\{\text{Rain "some day" in the future} | \text{Rain Monday \& Rain Tuesday}\}$

$\approx 0.25 + 0.15 = 0.40$

Example 4.10

An urn always contains 2 balls. Ball colors are red and blue.

At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the *same color*, and with probability 0.2 is the *opposite color*, as the ball it replaces.

If initially both balls are red, find the probability that the fifth ball selected is red.

$$X_n = \text{The number of red balls after the } n\text{th selection.} \quad n = 0, 1, 2, \ldots$$

Example 4.10 (cont.)

NOTE: $X_n \in \{0, 1, 2\}$, for all $n = 0, 1, 2, \ldots$

We then have:

$$P_{00} = 0.8, \qquad P_{01} = 0.2, \qquad P_{02} = 0.0$$

$$P_{10} = P\{\text{Red ball selected and replaced}\} = 0.5 \cdot 0.2 = 0.1$$
$$P_{11} = P\{\text{Any ball selected and not replaced}\} = 0.8$$
$$P_{12} = P\{\text{Blue ball selected and replaced}\} = 0.5 \cdot 0.2 = 0.1$$

$$P_{20} = 0.0, \qquad P_{21} = 0.2, \qquad P_{22} = 0.8$$

$$\boldsymbol{P} = \begin{bmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.8 & 0.1 \\ 0.0 & 0.2 & 0.8 \end{bmatrix}$$

Example 4.10 (cont.)

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{bmatrix}$$

P(Selection 5 is red)

$$= \sum_{i=0}^{2} P(\text{Selection 5 is red} | X_4 = i) \cdot P(X_4 = i | X_0 = 2)$$

$$= 0.00 \cdot P_{2,0}^4 + 0.50 \cdot P_{2,1}^4 + 1.00 \cdot P_{2,2}^4$$

$$= 0.50 \cdot 0.4352 + 0.4872 = 0.7048$$

Example 4.10 (cont.)

$$\boldsymbol{P}^{(30)} \approx \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \end{bmatrix}$$

P(Selection 31 is red)

$$= \sum_{i=0}^{2} P(\text{Selection 31 is red} | X_{30} = i) \cdot P(X_{30} = i | X_0 = 2)$$

$$= 0.00 \cdot P_{2,0}^{30} + 0.50 \cdot P_{2,1}^{30} + 1.00 \cdot P_{2,2}^{30}$$

$$\approx 0.50 \cdot 0.50 + 0.25 = 0.50$$

Example 4.11

Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns.

What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

$$X_n = \text{Number of nonempty urns after } n \text{ distributions} \quad n = 0, 1, 2 \ldots$$

Transition probabilities:

$$P_{i,i} = \frac{i}{8}, \qquad P_{i,i+1} = \frac{8-i}{8}, \quad i = 0, 1, \ldots, 8.$$

Example 4.11 (cont.)

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/8 & 6/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/8 & 5/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 4/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By calculating $\boldsymbol{P}^{(9)}$ we find the solution to the problem:

$$P(X_9 = 3 | X_0 = 0) = P_{03}^{9} = 0.00756.$$

However, there are some significant shortcuts here.

Example 4.11 (cont.)

The first transition of $\{X_n\}$ is deterministic (from state 0 to 1). Thus, we may instead consider the Markov chain $\{Y_n\}$ instead:

$$Y_n = X_{n+1}, \quad n = 0, 1, 2, \ldots,$$

with state space $\{1, 2, \ldots, 8\}$ and use that $P(X_9 = 3) = P(Y_8 = 3)$.

We can simplify the problem even further by letting:

$$Z_n = \min\{Y_n, 4\}, \quad n = 0, 1, 2, \ldots,$$

and use that $P(Y_8 = 3) = p(Z_8 = 3)$.

Example 4.11 (cont.)

The state space of the Markov chain $\{Z_n\}$ is $\{1, 2, 3, 4\}$ and its transition matrix is:

$$\boldsymbol{Q} = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.11 (cont.)

We first calculate:

$$\boldsymbol{Q}^{(4)} = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0.0000 & 0.0039 & 0.0952 & 0.9009 \\ 0.0000 & 0.0000 & 0.0198 & 0.9802 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

In order to find $\boldsymbol{Q}^{(8)}$, we use that:

$$\boldsymbol{Q}^{(8)} = \boldsymbol{Q}^{(4)} \cdot \boldsymbol{Q}^{(4)}$$

In particular:

$$P(X_9 = 3) = P(Z_8 = 3) = Q^8_{1,3} = \sum_{k=1}^{4} Q^4_{1,k} \cdot Q^4_{k,3}$$

$$= 0.0002 \cdot 0.2563 + 0.0256 \cdot 0.0952 + 0.2563 \cdot 0.0198 = 0.00756.$$

First-passage probabilities

Consider a Markov chain $\{X_n\}$ with state space $\mathcal{S}$, and let $\mathcal{A}$ be a non-empty proper subset of $\mathcal{S}$.

We want to calculate the following probability:

$$P(X_k \in \mathcal{A} \text{ for some } 1 \leq k \leq m | X_0 = i)$$

In order to analyze this we introduce:

$$N = \min\{n : X_n \in \mathcal{A}\},$$

where we let $N = \infty$ is $X_n \notin \mathcal{A}$ for all n.

NOTE: N represents the first time the Markov chain enters $\mathcal{A}$.

$$W_n = \begin{cases} X_n & \text{if } n < N \\ A & \text{if } n \geq N \end{cases}$$

NOTE: When $\{X_n\}$ enters $\mathcal{A}$, $\{W_n\}$ is absorbed in state A.

First-passage probabilities (cont.)

The transition probabilities of $\{W_n\}$, denoted $Q_{i,j}$, are given by:

$$Q_{i,j} = P_{i,j}, \qquad i \notin \mathcal{A}, j \notin \mathcal{A},$$

$$Q_{i,A} = \sum_{j \in \mathcal{A}} P_{i,j}, \quad i \notin \mathcal{A}, j \in \mathcal{A},$$

$$Q_{A,A} = 1.$$

We now have:

$$P(X_k \in \mathcal{A} \text{ for some } 1 \leq k \leq m | X_0 = i)$$
$$= P(W_m = A | X_0 = i)$$
$$= P(W_m = A | W_0 = i) = Q_{i,A}^m.$$

Example 4.12

In a sequence of independent flips of a fair coin, let N denote the number of flips until there is a run of 3 consecutive heads. We want to calculate:

$$P(N \leq 8) = ?$$
$$P(N = 8) = ?$$

To solve this problem we introduce a Markov chain $\{W_n\}$ with states $\mathcal{S} = \{0, 1, 2, 3\}$ defined relative to the sequence of coin flips as follows:

If $W_n = i < 3$, we currently are on a run of i consecutive heads, and that there has not yet been a run of 3 consecutive heads.

If $W_n = 3$, a run of 3 consecutive heads has occurred.

Example 4.12 (cont.)

The transition matrix of $\{W_n\}$ is:

$$\boldsymbol{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.12 (cont.)

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} 0.5000 & 0.2500 & 0.2500 & 0.0000 \\ 0.5000 & 0.2500 & 0.0000 & 0.2500 \\ 0.2500 & 0.2500 & 0.0000 & 0.5000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.4375 & 0.2500 & 0.1250 & 0.1875 \\ 0.3750 & 0.1875 & 0.1250 & 0.3125 \\ 0.2500 & 0.1250 & 0.0625 & 0.5625 \\ 0,0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

Example 4.12 (cont.)

$$\boldsymbol{P}^{(7)} = \begin{bmatrix} 0.3438 & 0.1875 & 0.1016 & 0.3672 \\ 0.2891 & 0.1563 & 0.0859 & 0.4688 \\ 0.1875 & 0.1016 & 0.0547 & 0.6563 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.3164 & 0.1719 & 0.0938 & 0.4180 \\ 0.2656 & 0.1445 & 0.0781 & 0.5117 \\ 0.1719 & 0.0938 & 0.0508 & 0.6836 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

Example 4.12 (cont.)

This implies that:

$$P(N \leq 8) = P^8_{0,3} = 0.4180$$

$$\begin{aligned} P(N = 8) &= P(N \leq 8) - P(N \leq 7) \\ &= P^8_{0,3} - P^7_{0,3} \\ &= 0.4180 - 0.3672 = 0.0508 \end{aligned}$$

Alternatively:

$$P(N = 8) = P^7_{0,2} \cdot 0.5 = 0.1016 \cdot 0.5 = 0.0508$$