STK2130 – Week 5

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Chapter 4.4 Long-Run Proportions and Limiting Probabilities

For pairs of states $i \neq j$ we let $f_{i,j}$ denote the probability that the Markov chain, starting in state i, will ever make a transition into state j:

$$\begin{aligned} f_{ij} &= P(X_n = j \text{ for some } n > 0 | X_0 = i) \\ &= P(\bigcup_{n=1}^{\infty} \{X_n = j\} | X_0 = i) \end{aligned}$$

We recall that if $i \rightarrow j$ if and only if $f_{ij} > 0$. We now show that:

Proposition (4.3)

If i is recurrent and $i \leftrightarrow j$, then $f_{ij} = 1$.

Proof of Proposition 4.3

Proof: Since $i \leftrightarrow j$ there exists an $n > 0$ such that $P_{ij}^n > 0$. We assume that n is the minimal integer with this property.

Moreover, since state i is recurrent, with probability one there exists an infinite sequence $0 = k_0 < k_1 < k_2 < \cdots$, such that $X_{k_r} = i$, $r = 0, 1, 2, \ldots$.

We then introduce:

$$Z = \min\{r \geq 0 : X_{k_r+n} = j\}$$

Then it is easy to verify that:

$$P(Z = z) = P_{ij}^n \cdot (1 - P_{ij}^n)^z, \quad z = 0, 1, 2, \ldots.$$

And from this it follows that:

$$1 \geq f_{ij} = P(\bigcup_{n=1}^{\infty} \{X_n = j\} | X_0 = i) \geq \sum_{z=0}^{\infty} P(Z = z) = 1.$$

Hence, we conclude $f_{ij} = 1$ $\blacksquare$

Positive and null recurrency

Assume that j is a recurrent state and introduce:

$$N_j = \min\{n > 0 : X_n = j\}$$

Thus, N_j is the number of steps until the Markov chain makes a transition into state j.

We then let:

$$m_j = E[N_j | X_0 = j]$$

That is, m_j is the expected number of steps until the Markov chain returns to state j given that it starts out in state j.

NOTE: Since j is recurrent, we know that $P(N_j < \infty | X_0 = j) = 1$.

Still, depending on the distribution of N_j, it may happen that $E[N_j | X_0 = j] = \infty$.

Positive and null recurrency (cont.)

Definition

If $m_j < \infty$, we say that j is positive recurrent.
If $m_j = \infty$, we say that j is null recurrent.

Let π_j be the long-run proportion of time the Markov chain is in state j:

$$\pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} I(X_r = j)$$

Proposition (4.4)

If the Markov chain is irreducible and recurrent, then for any initial state X_0, we have:

$$\pi_j = 1/m_j$$

NOTE: If $m_j = \infty$, then $\pi_j = 0$.

Proof of Proposition 4.4

Proof: Assume that $X_0 = i$, and introduce:

$$T_0 = \min\{r > 0 : X_r = j\}$$

$$T_1 = \min\{r > 0 : X_{T_0+r} = j\}$$

$$T_k = \min\{r > 0 : X_{T_0+\cdots+T_{k-1}+r} = j\}, \quad k = 2, 3, \ldots.$$

We then note:

- $P(T_0 < \infty) = f_{ij} = 1$ by Proposition 4.3.
- $T_1, T_2, \ldots$ are independent and identically distributed.
- $E[T_k] = m_j, \quad k = 1, 2, \ldots$.

Hence, by the strong law of large numbers:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_k = m_j \quad \text{with probability 1.}$$

Proof of Proposition 4.4 (cont.)

$T_0 + \sum_{k=1}^{n} T_k$ is the time the chain enters state j for the $(n+1)$st time.

The proportion of time the chain has been in state j at this point of time is:

$$\frac{\text{Number of times in } j}{\text{Total time}} = \frac{n+1}{T_0 + \sum_{k=1}^{n} T_k}$$

Hence, the long-run proportion is given by:

$$\pi_j = \lim_{n\to\infty} \frac{n+1}{T_0 + \sum_{k=1}^{n} T_k} = \lim_{n\to\infty} \frac{1}{\frac{T_0}{n+1} + \frac{n}{n+1} \cdot \frac{1}{n} \sum_{k=1}^{n} T_k} = \frac{1}{m_j}$$

NOTE: We have that $m_j < \infty$ if and only if $1/m_j > 0$.

Thus, state j is positive recurrent if and only if $\pi_j = 1/m_j > 0$.

Positive recurrence is a class property

Proposition (4.5)

If state i is positive recurrent and $i \leftrightarrow j$, then state j is positive recurrent as well.

Proof: Since i is positive recurrent, we know that $\pi_i > 0$. Moreover, since $i \leftrightarrow j$, there exists an $n > 0$ such that $P_{ij}^n > 0$.

From this it follows that:

$$\pi_j \geq \pi_i P_{ij}^n > 0.$$

Hence, state j is positive recurrent as well ■

Positive recurrence is a class property (cont.)

Corollary (4.5.1)

If state i is null recurrent and $i \leftrightarrow j$, then state j is null recurrent as well.

Proof: Assume that i is null recurrent and $i \leftrightarrow j$. If j is positive recurrent, Proposition 4.5 implies that i is positive recurrent as well. However, this contradicts the assumption $\blacksquare$

Corollary (4.5.2)

An irreducible finite state Markov chain must be positive recurrent.

Proof: By Proposition 4.5 all states in an irreducible are either positive recurrent or null recurrent. If all states are null recurrent, then $\pi_i = 0$ for all $i \in \mathcal{S}$. However, this is impossible if $|\mathcal{S}|$ is finite $\blacksquare$

Long-run proportion of states

We have that:

$$\pi_i P_{ij} = \text{Long-run proportion of transitions that go from } i \text{ to } j$$

Hence, by summing over all possible preceding states of j, we get:

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{ij}$$

Long-run proportion of states (cont.)

Theorem (4.1)

Consider an irreducible Markov chain. If the chain is positive recurrent, then the long-run proportions are the unique solution of the equations:

$$\pi_j = \sum_{i \in \mathcal{S}} \pi_i P_{ij}, \quad \text{for all } j \in \mathcal{S}$$

$$\sum_{j \in \mathcal{S}} \pi_j = 1$$

Moreover, if there is no solution of these linear equations, then the Markov chain is either transient or null recurrent, and $\pi_j = 0$ *for all* $j \in \mathcal{S}$.

Symmetric random walk

Consider a Markov chain with state space $\mathcal{S} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and where:

$$P_{i,i+1} = P_{i,i-1} = 1/2, \quad i \in \mathcal{S}.$$

By Example 4.19 we know that this chain is recurrent.

Assume that $X_0 = i$. Then by symmetry we must have $\pi_{i-1} = \pi_{i+1}$, and hence it follows by Theorem 4.1 that:

$$\pi_i = \pi_{i-1} \cdot \frac{1}{2} + \pi_{i+1} \cdot \frac{1}{2}$$

Since $\pi_{i-1} = \pi_{i+1}$, this implies that:

$$\pi_{i-1} = \pi_i = \pi_{i+1}.$$

Symmetric random walk (cont.)

Similarly it follows by Theorem 4.1 that:

$$\pi_{i+1} = \pi_i \cdot \frac{1}{2} + \pi_{i+2} \cdot \frac{1}{2}$$
$$\pi_{i-1} = \pi_i \cdot \frac{1}{2} + \pi_{i-2} \cdot \frac{1}{2}$$

Since $\pi_{i-1} = \pi_i = \pi_{i+1}$, this implies that:

$$\pi_{i-2} = \pi_i = \pi_{i+2}.$$

Continuing in the same way, we get that:

$$\pi_{i-k} = \pi_i = \pi_{i+k}, \quad k = 1, 2, \ldots$$

Since the initial state *i* was arbitrarily chosen, we conclude that the long-run proportions are the same for all states regardless of the initial state, and denote this common proportion by π.

Symmetric random walk (cont.)

If the chain is positive recurrent, it follows by Theorem 4.1 that:

$$\sum_{j \in \mathcal{S}} \pi_j = \pi \cdot \sum_{j \in \mathcal{S}} 1 = 1$$

However, $\sum_{j \in \mathcal{S}} 1 = \infty$, so this implies that $\pi = 0$.

Thus, we conclude that the chain is null recurrent.

Example 4.22

$$
\begin{aligned}
P\{\text{Rain tomorrow}|\text{Rain today}\} &= \alpha = 0.7 \\
P\{\text{Rain tomorrow}|\text{No rain today}\} &= \beta = 0.4
\end{aligned}
$$

$$
\boldsymbol{P} = \begin{bmatrix} \alpha & (1-\alpha) \\ \beta & (1-\beta) \end{bmatrix}
$$

In order to find the long-run proportion of rain (π_0) and not-rain (π_1), we solve the equations:

$$
\begin{aligned}
\pi_0 &= \alpha\pi_0 + \beta\pi_1 \\
\pi_1 &= (1-\alpha)\pi_0 + (1-\beta)\pi_1 \\
\pi_0 + \pi_1 &= 1.
\end{aligned}
$$

SOLUTION:

$$
\pi_0 = \frac{\beta}{1+\beta-\alpha} = \frac{4}{7}, \quad \pi_1 = \frac{1-\alpha}{1+\beta-\alpha} = \frac{3}{7}.
$$

Example 4.23 - Mood of an individual

0 = cheerful, 1 = so-so, 2 = glum.

$$\boldsymbol{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

In order to find the long-run proportions π_0, π_1 and π_2, we solve the equations:

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2 \\ \pi_1 &= 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2 \\ \pi_2 &= 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1. \end{aligned}$$

SOLUTION:

$$\pi_0 = \frac{21}{62} = 0.3387, \quad \pi_1 = \frac{23}{62} = 0.3710, \quad \pi_2 = \frac{18}{62} = 0.2903$$

Example 4.23 (cont.)

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.3446 & 0.3734 & 0.2820 \\ 0.3378 & 0.3706 & 0.2916 \\ 0.3330 & 0.3686 & 0.2984 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.3388 & 0.3710 & 0.2902 \\ 0.3387 & 0.3710 & 0.2903 \\ 0.3386 & 0.3709 & 0.2904 \end{bmatrix}$$

$$\boldsymbol{P}^{(16)} = \begin{bmatrix} 0.3387 & 0.3710 & 0.2903 \\ 0.3387 & 0.3710 & 0.2903 \\ 0.3387 & 0.3710 & 0.2903 \end{bmatrix}$$

Example 4.24 - Class mobility

0 = Upper class, 1 = Middle class, 2 = Lower class.

$$\boldsymbol{P} = \begin{bmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{bmatrix}$$

In order to find the long-run proportions π_0, π_1 and π_2, we solve the equations:

$$\begin{aligned} \pi_0 &= 0.45\pi_0 + 0.05\pi_1 + 0.01\pi_2 \\ \pi_1 &= 0.48\pi_0 + 0.70\pi_1 + 0.50\pi_2 \\ \pi_2 &= 0.07\pi_0 + 0.25\pi_1 + 0.49\pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1. \end{aligned}$$

SOLUTION:

$$\pi_0 = 0.0624, \quad \pi_1 = 0.6234, \quad \pi_2 = 0.3142$$

Example 4.24 (cont.)

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.0932 & 0.6199 & 0.2869 \\ 0.0623 & 0.6241 & 0.3136 \\ 0.0564 & 0.6229 & 0.3207 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.0635 & 0.6233 & 0.3132 \\ 0.0624 & 0.6234 & 0.3142 \\ 0.0622 & 0.6235 & 0.3144 \end{bmatrix}$$

$$\boldsymbol{P}^{(16)} = \begin{bmatrix} 0.0624 & 0.6234 & 0.3142 \\ 0.0624 & 0.6234 & 0.3142 \\ 0.0624 & 0.6234 & 0.3142 \end{bmatrix}$$

Example 4.25 - The Hardy-Weinberg Law

Two gene types: A and a

Three possible gene pairs: AA, aa, Aa.

In generation 0 we assume that the proportions of these gene pairs are respectively:

$$p_0 = \text{Proportion of } AA, \quad q_0 = \text{Proportion of } aa, \quad r_0 = \text{Proportion of } Aa$$

By conditioning on the gene pairs of a parent we get the following probabilities for one of the genes for a given child:

$$\begin{aligned} P(A) &= P(A|AA)p_0 + P(A|aa)q_0 + P(A|Aa)r_0 \\ &= 1 \cdot p_0 + 0 \cdot q_0 + \tfrac{1}{2} \cdot r_0 = p_0 + \tfrac{1}{2} \cdot r_0 \end{aligned}$$

$$\begin{aligned} P(a) &= P(a|AA)p_0 + P(a|aa)q_0 + P(a|Aa)r_0 \\ &= 0 \cdot p_0 + 1 \cdot q_0 + \tfrac{1}{2} \cdot r_0 = q_0 + \tfrac{1}{2} \cdot r_0 \end{aligned}$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

From this we get the proportions of the gene pairs in the generation 1:

$$
\begin{aligned}
p &= P(A) \cdot P(A) = (p_0 + \tfrac{1}{2} \cdot r_0)^2 \\
q &= P(a) \cdot P(a) = (q_0 + \tfrac{1}{2} \cdot r_0)^2 \\
r &= 2P(A)P(a) = 2 \cdot (p_0 + \tfrac{1}{2} \cdot r_0)(q_0 + \tfrac{1}{2} \cdot r_0)
\end{aligned}
$$

Hence, in the generation 1 the probabilities for the two gene types are:

$$
\begin{aligned}
P(A) &= p + \tfrac{1}{2} \cdot r \\
&= (p_0 + \tfrac{1}{2} \cdot r_0)^2 + (p_0 + \tfrac{1}{2} \cdot r_0)(q_0 + \tfrac{1}{2} \cdot r_0) \\
&= (p_0 + \tfrac{1}{2} \cdot r_0)[p_0 + \tfrac{1}{2} \cdot r_0 + q_0 + \tfrac{1}{2} \cdot r_0] \\
&= p_0 + \tfrac{1}{2} \cdot r_0
\end{aligned}
$$

$$
\begin{aligned}
P(a) &= q + \tfrac{1}{2} \cdot r \\
&= (q_0 + \tfrac{1}{2} \cdot r_0)^2 + (p_0 + \tfrac{1}{2} \cdot r_0)(q_0 + \tfrac{1}{2} \cdot r_0) \\
&= (q_0 + \tfrac{1}{2} \cdot r_0)[q_0 + \tfrac{1}{2} \cdot r_0 + p_0 + \tfrac{1}{2} \cdot r_0] \\
&= q_0 + \tfrac{1}{2} \cdot r_0
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

We now define:

$$X_n = \text{The gene pair of an } n\text{th generation child}, \quad n = 1, 2, \ldots$$

where the state space is $\mathcal{S} = \{AA, aa, Aa\}$.

The transition matrix for this chain is:

$$\boldsymbol{P} = \begin{bmatrix} p + r/2 & 0 & q + r/2 \\ 0 & q + r/2 & p + r/2 \\ p/2 + r/4 & q/2 + r/4 & p/2 + q/2 + r/2 \end{bmatrix}$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

To see this, we proceed as follows:

$$
\begin{aligned}
P(X_{n+1} &= AA | X_n = AA) \\
&= P(X_{n+1} = AA | X_n = AA, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = AA | X_n = AA, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = AA | X_n = AA, \text{ other parent is } Aa) \cdot r \\
&= 1 \cdot p + 0 \cdot q + \tfrac{1}{2} \cdot r = p + \frac{r}{2}
\end{aligned}
$$

$$P(X_{n+1} = aa | X_n = AA) = 0$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$
\begin{aligned}
P(X_{n+1} &= Aa | X_n = AA) \\
&= P(X_{n+1} = Aa | X_n = AA, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = Aa | X_n = AA, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = Aa | X_n = AA, \text{ other parent is } Aa) \cdot r \\
&= 0 \cdot p + 1 \cdot q + \tfrac{1}{2} \cdot r = q + \frac{r}{2}
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$P(X_{n+1} = AA | X_n = aa) = 0$$

$$\begin{aligned}
P(X_{n+1} &= aa | X_n = aa) \\
&= P(X_{n+1} = aa | X_n = aa, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = aa | X_n = aa, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = aa | X_n = aa, \text{ other parent is } Aa) \cdot r \\
&= 0 \cdot p + 1 \cdot q + \tfrac{1}{2} \cdot r = q + \frac{r}{2}
\end{aligned}$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$
\begin{aligned}
P(X_{n+1} &= Aa|X_n = aa) \\
&= P(X_{n+1} = Aa|X_n = aa, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = Aa|X_n = aa, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = Aa|X_n = aa, \text{ other parent is } Aa) \cdot r \\
&= 1 \cdot p + 0 \cdot q + \tfrac{1}{2} \cdot r = p + \frac{r}{2}
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$
\begin{aligned}
P(X_{n+1} = AA | X_n = Aa) \\
&= P(X_{n+1} = AA | X_n = Aa, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = AA | X_n = Aa, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = AA | X_n = Aa, \text{ other parent is } Aa) \cdot r \\
&= \tfrac{1}{2} \cdot p + 0 \cdot q + \tfrac{1}{4} \cdot r = \frac{p}{2} + \frac{r}{4}
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$
\begin{aligned}
P(X_{n+1} &= aa | X_n = Aa) \\
&= P(X_{n+1} = aa | X_n = Aa, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = aa | X_n = Aa, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = aa | X_n = Aa, \text{ other parent is } Aa) \cdot r \\
&= 0 \cdot p + \tfrac{1}{2} \cdot q + \tfrac{1}{4} \cdot r = \frac{q}{2} + \frac{r}{4}
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

$$
\begin{aligned}
P(X_{n+1} = Aa | X_n = Aa) \\
&= P(X_{n+1} = Aa | X_n = Aa, \text{ other parent is } AA) \cdot p \\
&+ P(X_{n+1} = Aa | X_n = Aa, \text{ other parent is } aa) \cdot q \\
&+ P(X_{n+1} = Aa | X_n = Aa, \text{ other parent is } Aa) \cdot r \\
&= \tfrac{1}{2} \cdot p + \tfrac{1}{2} \cdot q + \tfrac{1}{2} \cdot r = \frac{p}{2} + \frac{q}{2} + \frac{r}{2}
\end{aligned}
$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

We now assume that the distribution p, q, r is stable from generation to generation. This means that:

$$
\begin{aligned}
p &= P(A) \cdot P(A) = (p + \frac{r}{2})^2 \\
q &= P(a) \cdot P(a) = (q + \frac{r}{2})^2 \\
r &= 2P(A)P(a) = 2 \cdot (p + \frac{r}{2})(q + \frac{r}{2})
\end{aligned}
$$

We then claim that this implies that p, q, r also is the long-time distribution of the Markov chain with transition matrix P.

Since obviously $p + q + r = 1$, it is sufficient to verify that:

$$(p, q, r)P = (p, q, r)$$

Example 4.25 - The Hardy-Weinberg Law (cont.)

That is, we must check:

$$p(p + \frac{r}{2}) + r(\frac{p}{2} + \frac{r}{4}) = (p + \frac{r}{2})^2 = p$$

$$q(q + \frac{r}{2}) + r(\frac{q}{2} + \frac{r}{4}) = (q + \frac{r}{2})^2 = q$$

$$\begin{aligned} p(q + \frac{r}{2}) &+ q(p + \frac{r}{2}) + r(\frac{p}{2} + \frac{q}{2} + \frac{r}{2}) \\ &= p(q + \frac{r}{2}) + q(p + \frac{r}{2}) + \frac{r}{2}(p + \frac{r}{2} + q + \frac{r}{2}) \\ &= (p + \frac{r}{2})(q + \frac{r}{2}) + (q + \frac{r}{2})(p + \frac{r}{2}) \\ &= 2(p + \frac{r}{2})(q + \frac{r}{2}) = r \end{aligned}$$

Stationary probabilities

The long-run proportions π_j, $j \in \mathcal{S}$ are called the stationary probabilities of the Markov chain.

In fact if $P(X_0 = j) = \pi_j$, $j \in \mathcal{S}$, then $P(X_n = j) = \pi_j$, $j \in \mathcal{S}$, $n = 1, 2, \ldots$ as well.

To see this, we let $\pi_j^{(n)} = P(X_n = j)$, $j \in \mathcal{S}$, $n = 0, 1, 2, \ldots$. Moreover, let $\boldsymbol{\pi}^{(n)}$ denote the vector of $\pi_j^{(n)}$, $j \in \mathcal{S}$, and let $\boldsymbol{\pi}$ denote the vector of π_j, $j \in \mathcal{S}$.

Thus, $\boldsymbol{\pi} = \boldsymbol{\pi}^{(0)}$, and $\boldsymbol{\pi} = \boldsymbol{\pi} P$

By conditioning on X_{n-1} it follows that $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-1)} P$, $n = 1, 2, \ldots$.

Hence, $\boldsymbol{\pi}^{(1)} = \boldsymbol{\pi}^{(0)} P = \boldsymbol{\pi} P = \boldsymbol{\pi}$.

By induction this implies that $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi} P = \boldsymbol{\pi}$.

Bounded functions on the state space

Proposition (4.6)

Let $\{X_n\}$ be an irreducible Markov chain with stationary probabilities π_j, $j \in \mathcal{S}$, and let f be a bounded function on the state space. Then with probability 1*:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n) = \sum_{j\in\mathcal{S}} \pi_j f(j)$$

Proof: Let $a_j(N)$ be the amount of time the Markov chain spends in state j during the periods $1, \ldots, N$. Then we have:

$$\sum_{n=1}^{N} f(X_n) = \sum_{j\in\mathcal{S}} a_j(N) f(j)$$

Hence,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n) = \lim_{N\to\infty} \sum_{j\in\mathcal{S}} \frac{a_j(N)}{N} f(j) = \sum_{j\in\mathcal{S}} \pi_j f(j)$$

Example 4.29 - Car insurance

State space $\mathcal{S} = \{1, 2, 3, 4\}$ bonus classes. We let $f(j)$ denote the premium as a function of state, and assume that:

$$f(1) = 200, \quad f(2) = 250, \quad f(3) = 400, \quad f(4) = 600.$$

Transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0.6065 & 0.3033 & 0.0758 & 0.0144 \\ 0.6065 & 0.0000 & 0.3033 & 0.0902 \\ 0.0000 & 0.6065 & 0.0000 & 0.3935 \\ 0.0000 & 0.0000 & 0.6065 & 0.3935 \end{bmatrix}$$

The stationary distribution is found by solving $\pi = \pi P$ combined with the restriction that $\pi_1 + \cdots + \pi_4 = 1$, and we get:

$$\pi_1 = 0.3692, \quad \pi_2 = 0.2395, \quad \pi_3 = 0.2103, \quad \pi_4 = 0.1809$$

Average annual premium is then:

$$f(1) \cdot \pi_1 + f(2) \cdot \pi_2 + f(3) \cdot \pi_3 + f(4) \cdot \pi_4 = 326.375$$