STK2130 – Week 6

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Chapter 4.4.1. Limiting Probabilities

We consider a Markov chain with the following transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

From this it follows that:

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.575 & 0.425 \\ 0.567 & 0.433 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.571 & 0.429 \\ 0.571 & 0.429 \end{bmatrix}$$

Moreover:

$$\pi_0 = \frac{4}{7} \approx 0.571, \qquad \pi_1 = \frac{3}{7} \approx 0.429$$

From this example it is tempting to claim that:

$$\lim_{n \to \infty} P_{ij}^n = \pi_j, \quad \text{for all } i, j \in \mathcal{S}$$

COUNTER EXAMPLE:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If we solve the equations $\pi = \pi \boldsymbol{P}$ and $\pi_0 + \pi_1 = 1$, we get: $\pi_0 = \pi_1 = \frac{1}{2}$.

Chapter 4.4.1. Limiting Probabilities (cont.)

In this case we have for $n = 1, 2, \ldots$:

$$\boldsymbol{P}^{(2n)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{P}^{(2n+1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, $\lim_{n\to\infty} P_{ij}^{(n)}$ does not exist!

Definition

If a Markov chain can only return to a state in a multiple of $d > 1$ steps, it is said to be periodic. A Markov chain which is not periodic is said to be aperiodic. An irreducible, positive recurrent, aperiodic Markov chain is said to be ergodic.

Chapter 4.4.1. Limiting Probabilities (cont.)

Theorem

If a Markov chain with state space $\mathcal{S}$ is ergodic, then the limiting probabilities will always exist, and will not depend on the initial state, and we have:

$$\lim_{n\to\infty} P_{ij}^n = \pi_j, \qquad \textit{for all } i, j \in \mathcal{S}$$

"PROOF": Let $\alpha_j = \lim_{n\to\infty} P(X_n = j)$, $j \in \mathcal{S}$. We then have:

$$P(X_{n+1} = j) = \sum_{i\in\mathcal{S}} P(X_{n+1} = j | X_n = i) P(X_n = i) = \sum_{i\in\mathcal{S}} P_{ij} P(X_n = i)$$

$$\sum_{i\in\mathcal{S}} P(X_n = i) = 1$$

By letting n go to infinity, we then obtain:

$$\alpha_j = \sum_{i \in \mathcal{S}} P_{ij} \alpha_j$$

$$\sum_{i \in \mathcal{S}} \alpha_j = 1.$$

By Theorem 4.1 these equations have a unique solution, and thus we conclude that:

$$\alpha_j = \lim_{n \to \infty} P(X_n = j) = \pi_j, \qquad \text{for all } j \in \mathcal{S}.$$

Chapter 4.5.1. The Gambler's Ruin Problem

Potentially infinite sequence of independent identically distributed games.
$P(\text{Win one unit}) = p$, $P(\text{Lose one unit}) = q = 1 - p$.

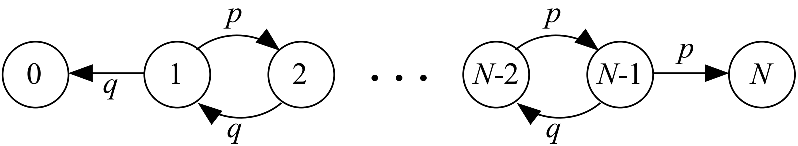
State space: $\mathcal{S} = \{0, 1, \ldots, N\}$ representing the player's fortune.

$$X_n = \text{The player's fortune after } n \text{ games}, \quad n = 0, 1, 2, \ldots$$

Transition probabilities:

$$\begin{aligned} P_{00} &= P_{NN} = 1 \\ P_{i,i+1} &= p, \quad i = 1, 2, \ldots, N - 1 \\ P_{i,i-1} &= q, \quad i = 1, 2, \ldots, N - 1 \end{aligned}$$

Classes: $\{0\}$ (recurrent), $\{1, 2, \ldots, N - 1\}$ (transient), $\{N\}$ (recurrent).



We then introduce:

$$P_i = P(\bigcup_{n=0}^{\infty} X_n = N | X_0 = i), \quad i = 0, 1, 2, \ldots, N.$$

Chapter 4.5.1. The Gambler's Ruin Problem (cont.)

By conditioning on X_1, we obtain:

$$P_i = pP_{i+1} + qP_{i-1}, \quad i = 1, 2, \ldots, N-1$$

Since $p + q = 1$, we may alternatively write:

$$pP_i + qP_i = pP_{i+1} + qP_{i-1}, \quad i = 1, 2, \ldots, N-1$$

or:

$$qP_i - qP_{i-1} = pP_{i+1} - pP_i, \quad i = 1, 2, \ldots, N-1$$

From this we get:

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), \quad i = 1, 2, \ldots, N-1$$

Chapter 4.5.1. The Gambler's Ruin Problem (cont.)

Since $P_0 = 0$ we get the following:

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = (\frac{q}{p})^2 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = (\frac{q}{p})^{i-1} P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = (\frac{q}{p})^{N-1} P_1$$

Chapter 4.5.1. The Gambler's Ruin Problem (cont.)

We then add the first $(i-1)$ equations:

$$(P_2 - P_1) + (P_3 - P_2) + \cdots + (P_i - P_{i-1})$$
$$= P_i - P_1 = \left[(\frac{q}{p}) + (\frac{q}{p})^2 + \cdots + (\frac{q}{p})^{i-1} \right] \cdot P_1$$

or equivalently:

$$P_i = \left[1 + (\frac{q}{p}) + (\frac{q}{p})^2 + \cdots + (\frac{q}{p})^{i-1} \right] \cdot P_1$$
$$= \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } \frac{q}{p} \neq 1 \\ iP_1 & \text{if } \frac{q}{p} = 1 \end{cases}$$

Chapter 4.5.1. The Gambler's Ruin Problem (cont.)

Now, we use that $(q/p) \neq 1$ if and only if $p \neq \frac{1}{2}$, and that $P_N = 1$.

CASE $p \neq \frac{1}{2}$

$$P_N = 1 = \frac{1 - (q/p)^N}{1 - (q/p)} P_1$$

Hence, in this case:

$$P_1 = \frac{1 - (q/p)}{1 - (q/p)^N}$$

CASE $p = \frac{1}{2}$

$$P_N = 1 = NP_1$$

Hence, in this case:

$$P_1 = \frac{1}{N}$$

Chapter 4.5.1. The Gambler's Ruin Problem (cont.)

By inserting the expression for P_1 into the formula for P_i we get:

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } p \neq \frac{1}{2} \\ iP_1 & \text{if } p = \frac{1}{2} \end{cases}$$

$$= \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

Note that if $p > \frac{1}{2}$ then $(q/p) < 1$, and hence $(q/p)^N \to 0$. Similarly, if $p < \frac{1}{2}$ then $(q/p) > 1$, and hence $(q/p)^N \to \infty$. Thus:

$$\lim_{N \to \infty} P_i = \begin{cases} 1 - (\frac{q}{p})^i & \text{if } p > \frac{1}{2} \\ 0 & \text{if } p \leq \frac{1}{2} \end{cases}$$

Example 4.30 - Penny flipping

We assume that $p = P(\text{Patty wins}) = 0.6$ and that $q = P(\text{Max wins}) = 0.4$.

Hence, $(q/p) = 0.4/0.6 = \frac{2}{3}$.

Moreover, we let X_n be the number of pennies owned by Patty after n plays.

CASE 1. $X_0 = 5$, $N = 5 + 10 = 15$

$$P_5 = \frac{1 - (\frac{2}{3})^5}{1 - (\frac{2}{3})^{15}} \approx 0.87$$

CASE 2. $X_0 = 10$, $N = 10 + 20 = 30$

$$P_{10} = \frac{1 - (\frac{2}{3})^{10}}{1 - (\frac{2}{3})^{30}} \approx 0.98$$

Drug testing

We consider two drug types and introduce:

$$\alpha_i = P(\text{A patient receiving drug number } i \text{ is cured}), \quad i = 1, 2.$$

α_1, α_2 are unknown, so we want to test whether $\alpha_1 > \alpha_2$ or vice versa.

EXPERIMENT: Pairs of patients are treated sequentially with one member of the pair receiving drug 1 and the other drug 2. The results for each pair are determined.

NB! Only pairs where the result for the patient who receives drug 1 is different from the result for the patient who receives drug 2 are included in the analysis.

The testing stops when the cumulative number of cures using one of the drugs exceeds the cumulative number of cures when using the other by some fixed predetermined number, M.

Drug testing (cont.)

Consider the *n*th pair where the result is different for the two drugs. Then:

$$p = P\{(\text{Drug 1 works}) \cap (\text{Drug 2 fails}) | \text{Different result}\}$$
$$= \frac{\alpha_1(1-\alpha_2)}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2}$$

$$q = P\{(\text{Drug 1 fails}) \cap (\text{Drug 2 works}) | \text{Different result}\}$$
$$= \frac{(1-\alpha_1)\alpha_2}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2}$$

We then introduce:

$X_n =$ The number of cured patients receiving drug 1 among the first n pairs

$-$ The number of cured patients receiving drug 2 among the first n pairs

Drug testing (cont.)

Then $\{X_n\}$ is a Markov chain with state space:

$$\mathcal{S} = \{-M, -(M-1), \ldots, -1, 0, 1, \ldots, (M-1), M\}$$

and transition probabilities:

$$\begin{aligned} P_{-M,-M} = P_{M,M} &= 1 \\ P_{i,i+1} &= p, \qquad i = -(M-1), \ldots, (M-1) \\ P_{i,i-1} &= q, \qquad i = -(M-1), \ldots, (M-1) \end{aligned}$$

Classes: $\{-M\}$ (recurrent), $\{-(M-1), \ldots, (M-1)\}$ (transient), $\{M\}$ (recurrent).

If the chain is absorbed in state M we conclude that $\alpha_1 > \alpha_2$, i.e., that drug 1 is the best drug.

If the chain is absorbed in state $-M$ we conclude that $\alpha_2 > \alpha_1$, i.e., that drug 2 is the best drug.

Drug testing (cont.)

Alternatively, let $Y_n = X_n + M$. Then $\{Y_n\}$ is a Markov chain with:

$$\mathcal{S} = \{0, 1, \ldots, (M-1), M, (M+1), \ldots, (2M-1), 2M\}$$

and transition probabilities:

$$\begin{aligned} P_{0,0} &= P_{2M,2M} = 1 \\ P_{i,i+1} &= p, \qquad i = 1, \ldots, (2M-1) \\ P_{i,i-1} &= q, \qquad i = 1, \ldots, (2M-1) \end{aligned}$$

Classes: $\{0\}$ (recurrent), $\{1, \ldots, (2M-1)\}$ (transient), $\{2M\}$ (recurrent).

If the chain is absorbed in state $2M$ we conclude that $\alpha_1 > \alpha_2$, i.e., that drug 1 is the best drug.

If the chain is absorbed in state 0 we conclude that $\alpha_2 > \alpha_1$, i.e., that drug 2 is the best drug.

Drug testing (cont.)

Assume that $X_0 = 0$ or equivalently that $Y_0 = X_0 + M = M$.

We then have:

$$P(\text{Test asserts that drug 1 is best} | X_0 = 0)$$

$$= P(\text{Test asserts that drug 1 is best} | Y_0 = M)$$

$$= \frac{1 - (q/p)^M}{1 - (q/p)^{2M}}$$

$$= \frac{1 - (q/p)^M}{(1 - (q/p)^M)(1 + (q/p)^M)}$$

$$= \frac{1}{1 + (q/p)^M}$$

Drug testing (cont.)

Similarly, we have:

$$
\begin{aligned}
&P(\text{Test asserts that drug 2 is best}|X_0 = 0) \\
&= P(\text{Test asserts that drug 2 is best}|\,Y_0 = M) \\
&= 1 - \frac{1-(q/p)^M}{1-(q/p)^{2M}} \\
&= 1 - \frac{1}{1+(q/p)^M} = \frac{1+(q/p)^M - 1}{1+(q/p)^M} \\
&= \frac{(q/p)^M}{1+(q/p)^M} = \frac{1}{1+(p/q)^M}
\end{aligned}
$$

Drug testing (cont.)

Assume that $\alpha_1 = 0.6$, $\alpha_2 = 0.4$ and $M = 5$. Thus, drug 1 is the best drug.

Then we have:

$$\alpha_1(1-\alpha_2) = 0.6^2 = 0.36, \qquad \alpha_2(1-\alpha_1) = 0.4^2 = 0.16.$$

Hence, we have:

$$p = \frac{\alpha_1(1-\alpha_2)}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2} = \frac{0.36}{0.36+0.16} = 0.6923$$

$$q = \frac{\alpha_2(1-\alpha_1)}{\alpha_1(1-\alpha_2) + (1-\alpha_1)\alpha_2} = \frac{0.16}{0.36+0.16} = 0.3077$$

Drug testing (cont.)

From this we get that:

$$P(\text{Test asserts that drug 1 is best}|X_0 = 0)$$

$$= P(\text{Test asserts that drug 1 is best}|Y_0 = 5)$$

$$= \frac{1}{1 + (q/p)^5} = \frac{1}{1 + (0.3077/0.6923)^5} = 0.9830$$

$$P(\text{Test asserts that drug 2 is best}|X_0 = 0)$$

$$= P(\text{Test asserts that drug 2 is best}|Y_0 = 5)$$

$$= \frac{1}{1 + (p/q)^5} = \frac{1}{1 + (0.6923/0.3077)^5} = 0.0170$$

Drug testing (cont.)

If we increase M to 10, we get that:

$$P(\text{Test asserts that } \text{drug 1 is best} | X_0 = 0)$$

$$= P(\text{Test asserts that } \text{drug 1 is best} | Y_0 = 10)$$

$$= \frac{1}{1 + (q/p)^{10}} = \frac{1}{1 + (0.3077/0.6923)^{10}} = 0.9997$$

$$P(\text{Test asserts that } \text{drug 2 is best} | X_0 = 0)$$

$$= P(\text{Test asserts that } \text{drug 2 is best} | Y_0 = 10)$$

$$= \frac{1}{1 + (p/q)^{10}} = \frac{1}{1 + (0.6923/0.3077)^{10}} = 0.0003$$

Chapter 4.6. Mean time spent in transient states

Consider a finite state Markov chain $\{X_n\}$ with state space $\mathcal{S}$, and with transient states $\mathcal{T} = \{1, 2, \ldots, t\} \subset \mathcal{S}$, and let the transition probabilities between the transient states be:

$$\boldsymbol{P}_T = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1t} \\ \vdots & \vdots & \vdots & \vdots \\ P_{t1} & P_{t2} & \cdots & P_{tt} \end{bmatrix}$$

NOTE: Since $\boldsymbol{P}_T$ is only a submatrix of the full matrix of transition probabilities, the row sums in $\boldsymbol{P}_T$ are less than 1.

We then introduce for all $i, j \in \mathcal{T}$:

$$s_{ij} = E[\text{Number of periods in state } j | X_0 = i]$$

$$\delta_{ij} = I(i = j)$$

Mean time spent in transient states (cont.)

By conditioning on the initial transition we get for all $i, j \in \mathcal{T}$:

$$s_{ij} = \delta_{ij} + \sum_{k \in \mathcal{S}} P_{ik} s_{kj} = \delta_{ij} + \sum_{k \in \mathcal{T}} P_{ik} s_{kj} \tag{1}$$

where we have used that $s_{kj} = 0$ if $k \in \mathcal{S} \setminus \mathcal{T}$.

We then let $\boldsymbol{I}$ be the identity matrix of size t, and let:

$$\boldsymbol{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1t} \\ \vdots & \vdots & \vdots & \vdots \\ s_{t1} & s_{t2} & \cdots & s_{tt} \end{bmatrix}$$

Then (1) can be written in matrix notation as:

$$\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{P}_T \boldsymbol{S}.$$

Mean time spent in transient states (cont.)

This last equation can be rewritten as:

$$\boldsymbol{S} - \boldsymbol{P}_T \boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{P}_T)\boldsymbol{S} = \boldsymbol{I}.$$

We then multiply both sides of the last equation by $(\boldsymbol{I} - \boldsymbol{P}_T)^{-1}$ and get:

$$\boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{P}_T)^{-1}$$

That is, we can find s_{ij} for all $i, j \in \mathcal{T}$ by inverting the matrix $(\boldsymbol{I} - \boldsymbol{P}_T)$.

Example 4.32

We consider the gambler's ruin problem with $p = 0.4$, $q = 0.6$ and $N = 7$, and we want to determine:

$$s_{3,5} = \text{The expected number of times the player has 5 units}$$
$$s_{3,2} = \text{The expected number of times the player has 2 units}$$

In this case we have $\mathcal{T} = \{1, 2, \ldots, 6\}$.

The transition probabilities for this Markov chain is:

$$\begin{aligned} P_{i,i} &= 1.0, \qquad && i \in \mathcal{S} \setminus \mathcal{T} \\ P_{i,i} &= 0.0, && i \in \mathcal{T} \\ P_{i,i+1} &= 0.4, && i \in \mathcal{T} \\ P_{i,i-1} &= 0.6, && i \in \mathcal{T} \end{aligned}$$

Example 4.32 (cont.)

$$\mathbf{P}_T = \begin{bmatrix} 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}$$

By inverting $(\mathbf{I} - \mathbf{P}_T)$, we get:

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1} = \begin{bmatrix} 1.6149 & 1.0248 & 0.6314 & 0.3691 & 0.1943 & 0.0777 \\ 1.5372 & 2.5619 & 1.5784 & 0.9228 & 0.4857 & 0.1943 \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ 1.2458 & 2.0763 & 2.6299 & 2.9990 & 1.5784 & 0.6314 \\ 0.9835 & 1.6391 & 2.0763 & 2.3677 & 2.5619 & 1.0248 \\ 0.5901 & 0.9835 & 1.2458 & 1.4206 & 1.5372 & 1.6149 \end{bmatrix}$$

Hence: $s_{3,5} = 0.9228$ and $s_{3,2} = 2.3677$.

Probability of transitions into transient states

For all $i, j \in \mathcal{T}$ we introduce:

$$f_{ij} = P(\text{At least one transition into state } j | X_0 = i)$$

Then we have:

$$\begin{aligned} s_{ij} &= E[\text{Periods in } j | X_0 = i, \text{At least one trans. into } j] f_{ij} \\ &+ E[\text{Periods in } j | X_0 = i, \text{No trans. into } j](1 - f_{ij}) \\ &= (\delta_{ij} + s_{jj}) f_{ij} + \delta_{ij}(1 - f_{ij}) \\ &= \delta_{ij} + f_{ij} s_{jj}. \end{aligned}$$

Hence, we find that:

$$f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}, \quad i, j \in \mathcal{T}.$$

Example 4.33

What is the probability that the gambler ever has a fortune of 1?

SOLUTION: We recall that:

$$\boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{P}_T)^{-1} = \begin{bmatrix} 1.6149 & 1.0248 & 0.6314 & 0.3691 & 0.1943 & 0.0777 \\ 1.5372 & 2.5619 & 1.5784 & 0.9228 & 0.4857 & 0.1943 \\ 1.4206 & 2.3677 & 2.9990 & 1.7533 & 0.9228 & 0.3691 \\ 1.2458 & 2.0763 & 2.6299 & 2.9990 & 1.5784 & 0.6314 \\ 0.9835 & 1.6391 & 2.0763 & 2.3677 & 2.5619 & 1.0248 \\ 0.5901 & 0.9835 & 1.2458 & 1.4206 & 1.5372 & 1.6149 \end{bmatrix}$$

and observe that $s_{3,1} = 1.4206$ and $s_{1,1} = 1.6149$.
Hence, we get that:

$$f_{3,1} = \frac{s_{3,1}}{s_{1,1}} = \frac{1.4206}{1.6149} = 0.8797.$$

Example 4.33 (cont.)

Alternatively, we consider the Markov chain $\{Y_n\}$ where $Y_n = X_n - 1$, and where we define 0 and 6 as absorbing states for $\{Y_n\}$.

Moreover, we let:

$$P_i = P(\bigcup_{n=0}^{\infty} Y_n = 6 | Y_n = i), \quad i = 1, 2, \ldots, 6.$$

We recall that:

$$P_i = \frac{1 - (q/p)^i}{1 - (q/p)^N} = \frac{1 - (0.6/0.4)^i}{1 - (0.6/0.4)^6}, \quad i = 1, 2, \ldots, 6.$$

Then it follows that:

$$f_{3,1} = 1 - P_{3-1} = 1 - \frac{1 - (0.6/0.4)^2}{1 - (0.6/0.4)^6} = 0.8797.$$