STK2130 – Chapter 10.1

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10.1 Brownian Motion

Consider a symmetric discrete-time random walk process $\{X_n : n \geq 0\}$ with state space $\mathcal{X} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and transition probabilities:

$$P_{i,i-1} = P_{i,i+1} = \frac{1}{2}, \quad i \in \mathcal{X}.$$

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We now generalize this model by considering general timesteps of length Δt, and general steps in states of length Δx, and define:

$$X(t) = \Delta x (X_1 + X_2 + \cdots + X_{\lfloor t/\Delta t \rfloor})$$

where $X_1, X_2, \ldots$ are independent with $P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$, $i = 1, 2, \ldots$, and where $\lfloor t/\Delta t \rfloor$ is the number of time steps at time t.

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The state space of this process is:

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Moreover, the process changes states for each $t \in \{\Delta t, 2\Delta t, 3\Delta t, \ldots\}$.

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Moreover, the process changes states for each $t \in \{\Delta t, 2\Delta t, 3\Delta t, \dots\}$.

NOTE: If we let $\Delta t = \Delta x = 1$, the process $\{X(t) : t \geq 0\}$ is essentially a symmetric random walk.

10.1 Brownian Motion (cont.)

Since $E[X_i] = 0$ and $\text{Var}[X_i] = E[X_i^2] = 1$, and

$$X(t) = \Delta x(X_1 + X_2 + \cdots + X_{\lfloor t/\Delta t \rfloor})$$

it follows that:

$$E[X(t)] = \Delta x \lfloor t/\Delta t \rfloor E[X_i] = 0$$

$$\text{Var}[X(t)] = (\Delta x)^2 \lfloor t/\Delta t \rfloor \, \text{Var}[X_i] = (\Delta x)^2 \lfloor t/\Delta t \rfloor$$

We now let $\Delta x = \sigma\sqrt{\Delta t}$, and consider the limit when Δt goes to zero:

$$\lim_{\Delta t \to 0} E[X(t)] = 0$$

$$\lim_{\Delta t \to 0} \text{Var}[X(t)] = \lim_{\Delta t \to 0} \sigma^2 \Delta t \lfloor t/\Delta t \rfloor = \sigma^2 t$$

10.1 Brownian Motion (cont.)

Definition (Brownian motion)

A Brownian motion is a stochastic process $\{X(t) : t \geq 0\}$ *where:*

(i) $X(0) = 0$

(ii) $\{X(t) : t \geq 0\}$ *has stationary and independent increments*

(iii) $X(t) \sim N(0, \sigma^2 t), \quad t > 0$

If $\sigma = 1$, $\{X(t) : t \geq 0\}$ *is called a standard Brownian motion.*

NOTE: If $\{Y(t) : t \geq 0\}$ is a Brownian motion, where $Y(t) \sim N(0, \sigma^2 t)$, then $\{X(t) : t \geq 0\}$, where $X(t) = Y(t)/\sigma$, for all $t \geq 0$ is a standard Brownian motion.

10.1 Brownian Motion (cont.)

NOTE: If $\{X(t) : t \geq 0\}$ is a standard Brownian motion, it follows that:

$$X(t+h) - X(t) \sim N(0, h)$$

As $h \to 0$, $\mathrm{Var}[X(t+h) - X(t)] \to 0$.

From this it can be shown that $X(t)$ is continuous with probability 1.

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On the other hand we have that:

$$\frac{X(t+h) - X(t)}{h} \sim N(0, h^{-1})$$

As $h \to 0$, $\mathrm{Var}[(X(t+h) - X(t))/h] \to \infty$.

From this it can be shown that $X(t)$ is nowhere differentiable with probability 1.

10.1 Brownian Motion (cont.)

We now assume that $\{X(t) : t \geq 0\}$ is a standard Brownian motion. Let $0 < t_1 < t_2 < \cdots < t_n$, let $X_i = X(t_i)$, $i = 1, 2, \ldots, n$, and let:

$$Y_1 = X_1, \quad Y_2 = X_2 - X_1, \quad Y_3 = X_3 - X_2, \quad \ldots \quad Y_n = X_n - X_{n-1}$$

Finally, we introduce the density of a normal distribution with mean 0 and variance v, denoted by f_v:

$$f_v(x) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

We want to determine the joint density of $X_1, \ldots, X_n$, which denote by $f_{\boldsymbol{t}}(x_1, \ldots, x_n)$, where $\boldsymbol{t} = (t_1, \ldots, t_n)$.

10.1 Brownian Motion (cont.)

Since $\{X(t) : t \geq 0\}$ has stationary and independent increments, it follows that $Y_1, \ldots, Y_n$ are independent and that:

$$Y_1 \sim N(0, t_1), \quad Y_2 \sim N(0, t_2 - t_1), \quad \ldots \quad Y_n \sim N(0, t_n - t_{n-1})$$

Hence, the joint density of $Y_1, \ldots, Y_n$ is given by:

$$f_{t_1}(y_1) \cdot f_{t_2 - t_1}(y_2) \cdots f_{t_n - t_{n-1}}(y_n)$$

The joint density of $X_1, \ldots, X_n$ is then obtained by transforming the Y_is to the X_is.

This is a simple linear transformation, and it is easy to verify that the Jacobian determinant of this transformation is 1. Thus, by the change of variable formula we get that:

$$f_{\boldsymbol{t}}(x_1, \ldots, x_n) = f_{t_1}(x_1) \cdot f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

10.1 Brownian Motion (cont.)

More specifically, the joint density has the form:

$$f_{\boldsymbol{t}}(x_1, \ldots, x_n) = C(\boldsymbol{t})e^{-(1/2)Q(x_1,\ldots,x_n)}$$

where $C(\boldsymbol{t})$ is a suitable normalizing constant, and where:

$$Q(x_1, \ldots, x_n) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}$$

NOTE: This formula is valid for any $n \geq 1$ and for any $0 < t_1 < \cdots < t_n$. Moreover, from this formula we can derive all possible conditional densities as well.

10.1 Brownian Motion (cont.)

EXAMPLE: Let $0 < t_1 < t_2$, and let $X_1 = X(t_1)$ and $X_2 = X(t_2)$. Then the joint density of X_1 and X_2 is:

$$f_{\boldsymbol{t}}(x_1, x_2) = C(\boldsymbol{t})e^{-(1/2)Q(x_1,x_2)}$$

where:

$$Q(x_1, x_2) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1}$$

The marginal densities of X_1 and X_2 are respectively:

$$f_{t_1}(x_1) = C(t_1)e^{-(1/2)(x_1^2/t_1)}$$

$$f_{t_2}(x_2) = C(t_2)e^{-(1/2)(x_2^2/t_2)}$$

10.1 Brownian Motion (cont.)

The conditional density of X_2 given $X_1 = x_1$ then becomes:

$$f_{X_2|X_1=x_1} = \frac{f_{\boldsymbol{t}}(x_1, x_2)}{f_{t_1}(x_1)} = \frac{C(\boldsymbol{t})e^{-(1/2)\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1}\right]}}{C(t_1)e^{-(1/2)\left[\frac{x_1^2}{t_1}\right]}}$$

$$= C(t_2|t_1)e^{-(1/2)\left[\frac{(x_2-x_1)^2}{t_2-t_1}\right]}$$

where the normalizing constant $C(t_2|t_1) = C(\boldsymbol{t})/C(t_1)$.

From this it follows that $(X_2|X_1 = x_1) \sim N(x_1, t_2 - t_1)$.

10.1 Brownian Motion (cont.)

In order to find the conditional density of X_1 given $X_2 = x_2$, we rewrite $Q(x_1, x_2)$ as follows:

$$Q(x_1, x_2) = \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1}$$

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$$= \left[\frac{1}{t_1} + \frac{1}{t_2 - t_1}\right] x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2$$

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$$= \left[\frac{1}{t_1} + \frac{1}{t_2 - t_1}\right] x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2$$

$$= \frac{t_2}{t_1(t_2 - t_1)} x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2$$

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$$= \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1}{t_2} x_2^2\right]$$

10.1 Brownian Motion (cont.)

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$$
\begin{aligned}
Q(x_1, x_2) &= \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} = \frac{x_1^2}{t_1} + \frac{x_2^2 - 2x_2x_1 + x_1^2}{t_2 - t_1} \\
&= \left[\frac{1}{t_1} + \frac{1}{t_2 - t_1}\right] x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2 \\
&= \frac{t_2}{t_1(t_2 - t_1)} x_1^2 - \frac{2x_2}{t_2 - t_1} x_1 + \frac{1}{t_2 - t_1} x_2^2 \\
&= \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1}{t_2} x_2^2\right] \\
&= \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1^2}{t_2^2} x_2^2 + \left(\frac{t_1}{t_2} - \frac{t_1^2}{t_2^2}\right) x_2^2\right]
\end{aligned}
$$

10.1 Brownian Motion (cont.)

$$Q(x_1, x_2) = \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1^2}{t_2^2} x_2^2 + \left(\frac{t_1}{t_2} - \frac{t_1^2}{t_2^2} \right) x_2^2 \right]$$

10.1 Brownian Motion (cont.)

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$$= \frac{t_2}{t_1(t_2 - t_1)} \left(x_1 - \frac{t_1}{t_2} x_2 \right)^2 + \frac{t_2}{t_1(t_2 - t_1)} \frac{t_1}{t_2} \left(1 - \frac{t_1}{t_2} \right) x_2^2$$

10.1 Brownian Motion (cont.)

$$Q(x_1, x_2) = \frac{t_2}{t_1(t_2 - t_1)} \left[x_1^2 - 2\frac{t_1 x_2}{t_2} x_1 + \frac{t_1^2}{t_2^2} x_2^2 + \left(\frac{t_1}{t_2} - \frac{t_1^2}{t_2^2} \right) x_2^2 \right]$$

$$= \frac{t_2}{t_1(t_2 - t_1)} \left(x_1 - \frac{t_1}{t_2} x_2 \right)^2 + \frac{t_2}{t_1(t_2 - t_1)} \frac{t_1}{t_2} \left(1 - \frac{t_1}{t_2} \right) x_2^2$$

$$= \frac{(x_1 - t_1 x_2 / t_2)^2}{t_1(t_2 - t_1)/t_2} + \frac{x_2^2}{t_2}$$

10.1 Brownian Motion (cont.)

The conditional density of X_1 given $X_2 = x_2$ then becomes:

$$f_{X_1|X_2=x_2} = \frac{f_{\boldsymbol{t}}(x_1, x_2)}{f_{t_2}(x_2)} = \frac{C(\boldsymbol{t})e^{-(1/2)\left[\frac{(x_1 - t_1 x_2/t_2)^2}{t_1(t_2-t_1)/t_2} + \frac{x_2^2}{t_2}\right]}}{C(t_2)e^{-(1/2)\left[\frac{x_2^2}{t_2}\right]}}$$

$$= C(t_1|t_2)e^{-(1/2)\left[\frac{(x_1 - t_1 x_2/t_2)^2}{t_1(t_2-t_1)/t_2}\right]}$$

where the normalizing constant $C(t_1|t_2) = C(\boldsymbol{t})/C(t_2)$.

From this it follows that $(X_1|X_2 = x_2) \sim N(\frac{t_1}{t_2}x_2, \frac{t_1}{t_2}(t_2 - t_1))$.

Example 10.1

Bicycle race with two competitors, A and B.

$Y(t)$ = A's lead in seconds after $100t$ percent of the race is finished

We model $\{Y(t) : 0 \leq t \leq 1\}$ as a Brownian motion process with variance parameter σ^2.

We let $t_1 = \frac{1}{2}$ and $t_2 = 1$. Moreover, we let $X_i = \frac{Y(t_i)}{\sigma}$, $i = 1, 2$.

We have shown that:

$$(X_2 | X_1 = x_1) \sim N(x_1, t_2 - t_1) = N(x_1, \tfrac{1}{2})$$

$$(X_1 | X_2 = x_2) \sim N(\frac{t_1}{t_2} x_2, \frac{t_1}{t_2}(t_2 - t_1)) = N(\tfrac{x_2}{2}, \tfrac{1}{4})$$

Example 10.1 (cont.)

(a) Find the probability that A wins given that A leads by σ seconds when $t = \frac{1}{2}$.

SOLUTION:

$$P(Y(t_2) > 0 | Y(t_1) = \sigma) = P(\frac{Y(t_2)}{\sigma} > 0 | \frac{Y(t_1)}{\sigma} = 1)$$

$$= P(X_2 > 0 | X_1 = 1)$$

$$= P(\frac{X_2 - 1}{\sqrt{1/2}} > \frac{0 - 1}{\sqrt{1/2}} | X_1 = 1)$$

$$= 1 - \Phi(\frac{-1}{\sqrt{1/2}}) = \Phi(\sqrt{2}) \approx 0.9213$$

Example 10.1 (cont.)

(b) Find the probability that A leads when $t = \frac{1}{2}$ given that A leads by σ seconds when $t = 1$.

SOLUTION:

$$
\begin{aligned}
P(Y(t_1) > 0 | Y(t_2) = \sigma) &= P(\frac{Y(t_1)}{\sigma} > 0 | \frac{Y(t_2)}{\sigma} = 1) \\
&= P(X_1 > 0 | X_2 = 1) \\
&= P(\frac{X_1 - \frac{1}{2}}{\sqrt{1/4}} > \frac{0 - \frac{1}{2}}{\sqrt{1/4}} | X_2 = 1) \\
&= 1 - \Phi(\frac{-\frac{1}{2}}{\sqrt{1/4}}) = \Phi(1) \approx 0.8413
\end{aligned}
$$