STK2130 – Lecture 6 - part 1

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Chapter 4.7 Branching Processes

Population with X_0 individuals, each able to produce offspring of the same kind during its lifetime.

$$P_j = P(\text{An individual produces } j \text{ new offspring}), \quad j = 0, 1, 2, \ldots$$

ASSUMPTIONS: $P_0 > 0$ and $P_j < 1$ for $j = 0, 1, 2, \ldots$

$$X_n = \text{Population size in the } n\text{th generation.}, \quad n = 0, 1, 2, \ldots$$

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ P_0 & P_1 & P_2 & P_3 & \cdots \\ P_0^2 & 2P_0 \cdot P_1 & \cdots & \cdots & \cdots \\ P_0^3 & 3P_0^2 \cdot P_1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Chapter 4.7 Branching Processes (cont.)

NOTE:

- Since $P_{00} = 1$, then 0 is a recurrent state.
- Since $P_0 > 0$, it follows that $P_{j0} = P_0^j > 0$.

  Hence, state j is transient for all $j > 0$.
- Any finite set of transient states $\{1, 2, \ldots, n\}$ will be visited only a finite number of times.

  Hence, since $P_0 > 0$, the population size converges to 0 or ∞ with probability 1.

Chapter 4.7 Branching Processes (cont.)

Assume that $X_0 = 1$, and let μ and σ^2 denote respectively the mean and the variance of the number of offspring of an individual. Then:

$$\mu = \sum_{j=0}^{\infty} jP_j,$$

$$\sigma^2 = \sum_{j=0}^{\infty} (j - \mu)^2 P_j.$$

We also let Z_r be the number of offspring from individual r in the $(n-1)$st generation. Hence:

$$X_n = \sum_{r=1}^{X_{n-1}} Z_r$$

$$E[X_n] = E[E[X_n \mid X_{n-1}]] \\ = E[E[\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1}]] \\ = E[X_{n-1}\mu] = \mu E[X_{n-1}]$$

Since we have assumed that $X_0 = 1$, it follows by induction that:

$$E[X_n] = \mu^n.$$

To find $\mathrm{Var}[X_n]$ we use that:

$$\mathrm{Var}[X_n] = E[\mathrm{Var}(X_n \mid X_{n-1})] + \mathrm{Var}[E(X_n \mid X_{n-1})] \\ = E[\mathrm{Var}(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1})] + \mathrm{Var}[E(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1})]$$

$$= E[\mathrm{Var}(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1})] + \mathrm{Var}[E(\sum_{r=1}^{X_{n-1}} Z_r \mid X_{n-1})]$$

$$= E[X_{n-1}\sigma^2] + \mathrm{Var}[X_{n-1}\mu] = \sigma^2\mu^{n-1} + \mu^2 \,\mathrm{Var}[X_{n-1}]$$

$$= \sigma^2\mu^{n-1} + \mu^2(\sigma^2\mu^{n-2} + \mu^2 \,\mathrm{Var}[X_{n-2}])$$

$$= \sigma^2\mu^{n-1}(1+\mu) + \mu^4 \,\mathrm{Var}[X_{n-2}]$$

$$= \sigma^2\mu^{n-1}(1+\mu+\mu^2) + \mu^6 \,\mathrm{Var}[X_{n-3}]$$

$$\ldots$$

$$= \sigma^2\mu^{n-1}(1+\mu+\cdots+\mu^{n-1}) + \mu^{2n}\cdot \mathrm{Var}[X_0]$$

Hence, since $\mathrm{Var}[X_0] = 0$, we get:

$$\mathrm{Var}[X_n] = \begin{cases} \sigma^2\mu^{n-1}(\frac{1-\mu^n}{1-\mu}), & \text{if } \mu \neq 1 \\ n\sigma^2, & \text{if } \mu = 1 \end{cases}$$

Chapter 4.7 Branching Processes (cont.)

We then consider the probability that the population eventually dies out:

$$\pi_0 = \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1)$$

We first note that:

$$\mu^n = E[X_n] = E[X_n \mid X_0 = 1] = \sum_{j=1}^{\infty} j \cdot P(X_n = j \mid X_0 = 1)$$

$$\geq \sum_{j=1}^{\infty} 1 \cdot P(X_n = j \mid X_0 = 1) = 1 - P(X_n = 0 \mid X_0 = 1)$$

Hence, it follows that $P(X_n = 0 | X_0 = 1) \geq 1 - \mu^n$. If $\mu < 1$ this implies that:

$$1 \geq \pi_0 = \lim_{n \to \infty} P(X_n = 0 \mid X_0 = 1) \geq 1 - \lim_{n \to \infty} \mu^n = 1 - 0 = 1.$$

Thus, if $\mu < 1$, we must have $\pi_0 = 1$.

In general we have:

$$\pi_0 = P(\text{The population dies out}) \tag{1}$$

$$= \sum_{j=0}^{\infty} P(\text{The population dies out} \mid X_1 = j) P_j$$

$$= \sum_{j=0}^{\infty} \pi_0^j P_j$$

It can be shown that π_0 is the smallest positive number that satisfies (1).

NOTE 1: Since $\sum_{j=0}^{\infty} P_j = 1$, we see that $\pi_0 = 1$ is always a solution to (1).

NOTE 2: If $P_j = 0$ for all $j \geq 2$, this implies that $\mu < 1$, and hence, $\pi_0 = 1$. Thus, in the following we assume that $P_j > 0$ for at least one $j \geq 2$

We then introduce the following functions:

$$\phi(z) = \sum_{j=0}^{\infty} z^j P_j, \quad \ell(z) = z,$$

and note that a solution z to the equation (1) is found by solving $\phi(z) = \ell(z)$.

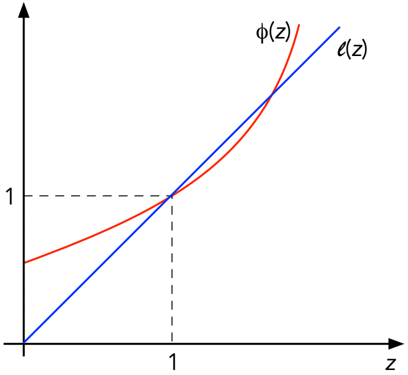
We observe that:

$$\phi(0) = \sum_{j=0}^{\infty} 0^j P_j = P_0 > 0, \qquad \phi(1) = \sum_{j=0}^{\infty} 1^j P_j = 1,$$

$$\phi'(z) = \sum_{j=1}^{\infty} j \cdot z^{j-1} P_j, \qquad \phi'(1) = \sum_{j=1}^{\infty} jP_j = \mu,$$

$$\phi''(z) = \sum_{j=2}^{\infty} j(j-1) \cdot z^{j-2} P_j, \qquad \phi''(z) > 0 \text{ for all } z > 0.$$

CASE 1. $\phi'(1) = \mu < 1$

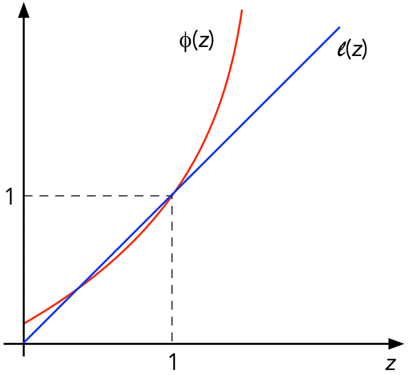


In this case $\phi(z) = \ell(z)$ for $z = 1$ and some $z > 1$.

The smallest positive number that satisfies (1) is $\pi_0 = 1$.

Chapter 4.7 Branching Processes (cont.)

CASE 2. $\phi'(1) = \mu > 1$

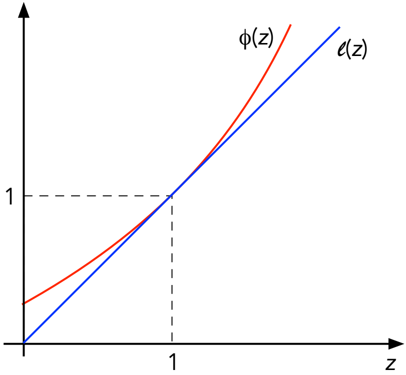


In this case $\phi(z) = \ell(z)$ for $z = 1$ and some $0 < z < 1$.

The smallest positive number that satisfies (1) is $\pi_0 < 1$.

Chapter 4.7 Branching Processes (cont.)

CASE 3. $\phi'(1) = \mu = 1$



In this case $\phi(z) = \ell(z)$ for $z = 1$ only.

The only positive number that satisfies (1) is $\pi_0 = 1$.

Chapter 4.7 Branching Processes (cont.)

CONCLUSION:

- If $\mu \leq 1$, then $\pi_0 = P(\text{The population dies out}) = 1$.
- If $\mu > 1$, then $\pi_0 = P(\text{The population dies out}) < 1$.

Example 4.34

Assume that $P_0 = \frac{1}{2}$, $P_1 = \frac{1}{4}$ and $P_2 = \frac{1}{4}$. Find π_0.

SOLUTION:

$$\begin{aligned}\mu &= 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 \\ &= 0 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{4} = \tfrac{3}{4} < 1.\end{aligned}$$

Hence, we must have $\pi_0 = 1$.

Example 4.35

Assume that $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{4}$ and $P_2 = \frac{1}{2}$. Find π_0.

SOLUTION:

$$\mu = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2$$
$$= 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{2} = \tfrac{5}{4} > 1.$$

In order to find π_0 we solve (1), which in this case becomes:

$$\pi_0 = \pi_0^0 P_0 + \pi_0^1 P_1 + \pi_0^2 P_2 = \tfrac{1}{4} + \tfrac{1}{4}\pi_0 + \tfrac{1}{2}\pi_0^2.$$

or equivalently:

$$2\pi_0^2 - 3\pi_0 + 1 = 2(\pi_0 - 1)(\pi_0 - \tfrac{1}{2}) = 0$$

Hence, the smallest positive number that satisfies (1) is $\pi_0 = \frac{1}{2}$.