STK2130 – Chapter 6.5

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6.5 Limiting Probabilities

Let $\{X(t) : t \geq 0\}$ be continuous-time Markov chain with state space $\mathcal{X}$ and transition probabilities $P_{ij}(t)$, $t \geq 0$, $i, j \in \mathcal{X}$.

The limiting distribution of this chain, denoted by π_j, is defined by:

$$\pi_j = \lim_{t \to \infty} P_{ij}(t), \quad j \in \mathcal{X},$$

assuming that the limit exists.

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assuming that the limit exists.

Note that if π_j exists, we must have:

$$\begin{aligned} \lim_{t \to \infty} P'_{ij}(t) &= \lim_{t \to \infty} \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} \\ &= \lim_{h \to 0} \lim_{t \to \infty} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \frac{\pi_j - \pi_j}{h} = 0. \end{aligned}$$

6.5 Limiting Probabilities (cont.)

To determine the limiting distribution, we use Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

By taking the limit on both sides when t goes to infinity, we get:

$$0 = \lim_{t \to \infty} P'_{ij}(t) = \lim_{t \to \infty} \left[\sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j \right]$$

$$= \sum_{k \in \mathcal{X} \setminus j} \pi_k q_{kj} - \pi_j v_j, \quad j \in \mathcal{X}.$$

Combined with the equation $\sum_{j \in \mathcal{X}} \pi_j = 1$, we can determine the limiting distribution.

6.5 Limiting Probabilities (cont.)

In the case where $\mathcal{X} = \{1, \ldots, n\}$ we introduce:

$$\boldsymbol{R} = \begin{bmatrix} -v_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ q_{2,1} & -v_2 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & -v_3 & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & -v_n \end{bmatrix}$$

and let $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$. Then the equations:

$$\sum_{k \in \mathcal{X} \setminus j} \pi_k q_{kj} - \pi_j v_j = 0, \quad j \in \mathcal{X}.$$

can be written as:

$$\boldsymbol{\pi} \boldsymbol{R} = \boldsymbol{0}.$$

where $\boldsymbol{0} = (0, \ldots, 0)$.

6.5 Limiting Probabilities (cont.)

The limiting distribution for continuous-time Markov chains is found by using the following equations:

$$\boldsymbol{\pi} \boldsymbol{R} = \boldsymbol{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

6.5 Limiting Probabilities (cont.)

The limiting distribution for continuous-time Markov chains is found by using the following equations:

$$\boldsymbol{\pi} \boldsymbol{R} = \boldsymbol{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

We compare this to the equations we use for discrete-time Markov chains:

$$\boldsymbol{\pi} \boldsymbol{P} = \boldsymbol{\pi}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

or equivalently:

$$\boldsymbol{\pi}(\boldsymbol{P} - \boldsymbol{I}) = \boldsymbol{0}, \qquad \sum_{j \in \mathcal{X}} \pi_j = 1$$

where $\boldsymbol{P}$ denotes the matrix of transition probabilities for the chain.

6.5 Limiting Probabilities (cont.)

NOTE: In order to determine the limiting distribution we used Kolmogorov's forward equations. What about Kolmogorov's backward equations?

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

By taking the limit on both sides when t goes to infinity, we then get:

$$0 = \lim_{t \to \infty} P'_{ij}(t) = \lim_{t \to \infty} \left[\sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \right]$$

$$= \sum_{k \in \mathcal{X} \setminus j} q_{kj} \pi_j - v_j \pi_j = \pi_j \left[\sum_{k \in \mathcal{X} \setminus j} q_{kj} - v_j \right] = 0.$$

Thus, in this case we do not get any non-trivial equations!!

6.5 Limiting Probabilities (cont.)

When the limiting probabilities exist, we say that the chain is ergodic.

Necessary and sufficient conditions for the existence of the limiting distribution are:

- All states of the Markov chain communicate in the sense that starting in state i there is a positive probability of ever being in state j, for all $i, j \in \mathcal{X}$.
- The Markov chain is positive recurrent in the sense that, starting in any state, the mean time to return to that state is finite.

If these conditions hold, the limiting probabilities exist and satisfy the derived equations.

In addition, the probability π_j also has the interpretation of being the long-run proportion of time that the process is in state j.

6.5 Limiting Probabilities (cont.)

It is often useful to write the equations for the limiting distribution in the following form:

$$\sum_{k \in \mathcal{X} \setminus j} \pi_k q_{kj} = \pi_j v_j, \quad j \in \mathcal{X}$$

This representation can be interpreted as follows:

- The left-hand side of the equation is the rate at which the process enters state j
- The right-hand side of the equation is the rate at which the process leaves state j
- These equations, referred to as the balance equations, state that these rates are the same for all states $j \in \mathcal{X}$.

Limiting distribution of a birth and death process

For a birth and death process the balance equations are as follows:

State	Leave rate	=	Enter rate
0	$\lambda_0 \pi_0$	=	$\mu_1 \pi_1$
1	$(\lambda_1 + \mu_1)\pi_1$	=	$\mu_2 \pi_2 + \lambda_0 \pi_0$
2	$(\lambda_2 + \mu_2)\pi_2$	=	$\mu_3 \pi_3 + \lambda_1 \pi_1$
...	...		...
n	$(\lambda_n + \mu_n)\pi_n$	=	$\mu_{n+1} \pi_{n+1} + \lambda_{n-1} \pi_{n-1}$

We observe that the left-hand side of the equation for state 1 contains the term $\mu_1 \pi_1$, while the right-hand side of the equation for state 1 contains the term $\lambda_0 \pi_0$.

By the equation for state 0 these two terms are equal, and thus we may remove them so that the equation for state 1 becomes:

$$\lambda_1 \pi_1 = \mu_2 \pi_2$$

Limiting distribution of a birth and death process (c.)

By repeated use of this argument we obtain the following simplified set of equations:

$$
\begin{aligned}
\lambda_0\pi_0 &= \mu_1\pi_1 \\
\lambda_1\pi_1 &= \mu_2\pi_2 \\
\lambda_2\pi_2 &= \mu_3\pi_3 \\
&\cdots\cdots \\
\lambda_n\pi_n &= \mu_{n+1}\pi_{n+1} \\
&\cdots\cdots
\end{aligned}
$$

Limiting distribution of a birth and death process (c.)

We can then express $\pi_1, \pi_2, \ldots$ in terms of π_0 as follows:

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$

$$\pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0$$

$$\pi_3 = \frac{\lambda_2}{\mu_3}\pi_2 = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}\pi_0$$

$$\vdots$$

$$\pi_n = \frac{\lambda_{n-1}}{\mu_n}\pi_{n-1} = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1}\pi_0$$

Limiting distribution of a birth and death process (c.)

Nest step is to determine π_0 by using that the limiting probabilities must add up to one:

$$1 = \pi_0 + \pi_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1}$$

Solving this equation with respect to π_0 yields:

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1}}$$

For $n \geq 1$ we have:

$$\pi_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1\left(1 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1}\right)}$$

Limiting distribution of a birth and death process (c.)

NOTE: A necessary and sufficient condition for the limiting distribution to exist is that:

$$\sum_{n=1}^{\infty} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_1\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_2\mu_1} < \infty$$

Example 6.14 Assume that $\lambda_i = \lambda$, $i = 0, 1, 2, \ldots$ and that $\mu_i = \mu$, $i = 1, 2, \ldots$.

Then the limiting distribution exists if and only if:

$$\sum_{n=1}^{\infty} \frac{\lambda^n}{\mu^n} < \infty$$

which holds if and only if $\lambda < \mu$.

Limiting distribution of a birth and death process (c.)

Note that by using the formula for an infinite geometric series and assuming $\lambda < \mu$ we have:

$$1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n = \sum_{n=0}^{\infty} (\lambda/\mu)^n = \frac{1}{1 - \lambda/\mu} = (1 - \lambda/\mu)^{-1}$$

Thus, the limiting distribution can be written as:

$$\begin{aligned} \pi_n &= \frac{(\lambda/\mu)^n}{1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n} \\ &= \frac{(\lambda/\mu)^n}{(1 - \lambda/\mu)^{-1}} \\ &= (\lambda/\mu)^n \cdot (1 - \lambda/\mu), \quad n \geq 0. \end{aligned}$$

Example 6.1 – A Shoe Shine Shop

A Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1, 2\}$ where:

- State 0. No customer
- State 1. Customer in chair 1 (clean and polish)
- State 2. Customer in chair 2 (polish is buffed)

$X(s) = 0$: In this state customers arrive in accordance to a Poisson process with rate λ. The time spent in this state is $T_0 \sim exp(\lambda)$. Then the process transits to state 1 with probability $Q_{01} = 1$.

$X(t) = 1$: The time spent in this state is $T_1 \sim exp(\mu_1)$. Then the process transits to state 2 with probability $Q_{12} = 1$.

$X(u) = 2$: The time spent in this state is $T_2 \sim exp(\mu_2)$. Then the process transits to state 0 with probability $Q_{20} = 1$, and then the process repeats the same cycle.

Example 6.15 – A Shoe Shine Shop

State	Leave rate	=	Enter rate
0	$\lambda\pi_0$	=	$\mu_2\pi_2$
1	$\mu_1\pi_1$	=	$\lambda\pi_0$
2	$\mu_2\pi_2$	=	$\mu_1\pi_1$

We can then express π_1, π_2 in terms of π_0 as follows:

$$\pi_1 = \frac{\lambda}{\mu_1}\pi_0, \qquad \pi_2 = \frac{\lambda}{\mu_2}\pi_0$$

Since $\pi_0 + \pi_1 + \pi_2 = 1$, we get the following equation for π_0:

$$\pi_0 \left[1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}\right] = \pi_0 \frac{\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1}{\mu_1\mu_2} = 1$$

Example 6.16 – A Shoe Shine Shop (cont.)

From this it follows that:

$$\pi_0 = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \lambda(\mu_2 + \mu_1)}$$

$$\pi_1 = \frac{\lambda}{\mu_1}\pi_0 = \frac{\lambda\mu_2}{\mu_1\mu_2 + \lambda(\mu_2 + \mu_1)}$$

$$\pi_2 = \frac{\lambda}{\mu_2}\pi_0 = \frac{\lambda\mu_1}{\mu_1\mu_2 + \lambda(\mu_2 + \mu_1)}$$

Stationary probabilities

Assume that $P(X(0) = j) = \pi_j, \ j \in \mathcal{X}$, and let $t > 0$. Then we have:

$$
\begin{aligned}
P(X(t) = j) &= \sum_{k \in \mathcal{X}} P(X(t) = j | X(0) = k) P(X(0) = k) \\
&= \sum_{k \in \mathcal{X}} P_{kj}(t) \pi_k \\
&= \sum_{k \in \mathcal{X}} P_{kj}(t) \lim_{s \to \infty} P_{ik}(s) \\
&= \lim_{s \to \infty} \sum_{k \in \mathcal{X}} P_{kj}(t) P_{ik}(s) \\
&= \lim_{s \to \infty} P_{ij}(t + s) = \pi_j
\end{aligned}
$$