STK2130 – Chapter 7.1

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7.1 Renewal theory – Introduction

Definition

Let $\{N(t) : t \geq 0\}$ *be a counting process and let* X_n *denote the nth interarrival time, i.e., the time between the* $(n-1)$*st and the nth event of this process,* $n = 1, 2, \ldots$.

If $X_1, X_2, \ldots$ *are independent and identically distributed, then* $\{N(t) : t \geq 0\}$ *is said to be a renewal process.*

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7.1 Renewal theory – Introduction (cont.)

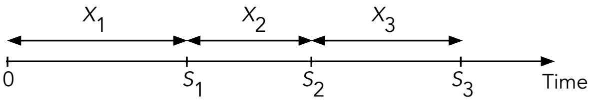

Figure: Renewal and interarrival times

For a renewal process the events are referred to as renewals.

If $\{N(t) : t \geq 0\}$ is a renewal process with interarrival times $X_1, X_2, \ldots$ we let:

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad n = 1, 2, \ldots$$

That is, $S_1 = X_1$ is the time of the first renewal, $S_2 = X_1 + X_2$ is the time of the second renewal. In general S_n is the time of the nth renewal.

7.1 Renewal theory – Introduction (cont.)

We denote the cumulative distribution function of the interarrival times by F, and assume that:

$$F(0) = P(X_n = 0) < 1, \quad \text{and} \quad \lim_{t \to \infty} F(t) = P(X_n < \infty) = 1.$$

We also assume that $E[X_n] = \mu > 0$.

We now show that: $N(t) < \infty$ for all t with probability 1.

By the strong law of large numbers we have with probability 1 that:

$$\lim_{n \to \infty} \frac{1}{n} S_n = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu$$

Since $\mu > 0$, this implies that $\lim_{n \to \infty} S_n = \infty$. Hence, for a given t, $S_n < t$ can only occur for a finite number of values of n, and so with probability 1:

$$N(t) = \max\{n : S_n \leq t\} < \infty$$

7.1 Renewal theory – Introduction (cont.)

We then show that: $N(\infty) = \lim_{t \to \infty} N(t) = \infty$ with probability 1.

This follows since:

$$\begin{aligned} P(N(\infty) < \infty) &= P(S_n = \infty \text{ for some } n) \\ &= P(X_n = \infty \text{ for some } n) \\ &= P\left(\bigcup_{n=1}^{\infty} X_n = \infty\right) \\ &\leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0 \end{aligned}$$

NOTE: The last equality follows since we have assumed that $P(X_n < \infty) = 1$.