STK2130 – Lecture 1

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Section 4.1 – Introduction

Discrete Time Stochastic Processes

- A *discrete time stochastic process* is a sequence $\{X_0, X_1, X_2, \ldots\}$ of random variables with values in some set $\mathcal{S}$.
- The process is sometimes denoted by $\{X_n : n \geq 0\}$, or by $\{X_n\}$.
- The index n is interpreted as *time*, and is a non-negative integer.
- If $\mathcal{S}$ is finite or countable, the process has a *discrete* (state) space.
- The elements of $\mathcal{S}$ are referred to as *states*. Thus, X_n is the state of the process at time n, $n = 0, 1, 2, \ldots$

In simple cases $X_0, X_1, \ldots$ are independent. However, in this course we will focus on cases where $X_0, X_1, \ldots$ are dependent.

Markov Chains

A discrete time, discrete space stochastic process $\{X_0, X_1, X_2, \ldots\}$ is called a time homogenuous Markov chain if the transition probabilities satisfy the following:

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P(X_{n+1} = j | X_n = i) = P_{ij}$$

for all states $i_0, i_1, \ldots, i_{n-1}, i, j$ and all $n \geq 0$.

NOTE: The Markov assumption implies that given X_n, the next state, X_{n+1}, is independent of all previous states $X_0, X_1, \ldots, X_{n-1}$. The process is time homogenuous since P_{ij} is independent of n.

One-step transition probability matrix:

$$\boldsymbol{P} = [P_{ij}]_{i,j \in \mathcal{S}}$$

where $P_{ij} \geq 0$ for all $i, j \in \mathcal{S}$, and $\sum_{j \in \mathcal{S}} P_{ij} = 1$ for all $i \in \mathcal{S}$.

Example 4.1 - Forcasting the Weather

Consider a Markov chain with state space $\mathcal{S} = \{0, 1\}$, where 0 represents rain and 1 represents no rain.

We assume that:

- If it rains today, then it will rain tomorrow with probability α
- If it does not rain today, then it will rain tomorrow with probability β.

Mathematically, this can be expressed as follows:

$$P_{0,0} = P(X_{n+1} = 0 | X_n = 0) = \alpha, \quad n = 0, 1, 2 \ldots$$

$$P_{1,0} = P(X_{n+1} = 0 | X_n = 1) = \beta, \quad n = 0, 1, 2 \ldots$$

This implies that we also have:

$$P_{0,1} = P(X_{n+1} = 1 | X_n = 0) = 1 - \alpha, \quad n = 0, 1, 2 \ldots$$

$$P_{1,1} = P(X_{n+1} = 1 | X_n = 1) = 1 - \beta, \quad n = 0, 1, 2 \ldots$$

Example 4.1 - Forcasting the Weather (cont.)

NOTE 1: We implicitly assume that the transition probabilities do not change over time, i.e., that the process is time-homogenous.

NOTE 2: Since we have assumed that $\{X_n\}$ is a Markov chain, this implies that given the weather today, the weather tomorrow is independent of the weather in past days.

The one-step transition probability matrix for $\{X_n\}$ is given by:

$$\boldsymbol{P} = \begin{bmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{bmatrix} = \begin{bmatrix} \alpha & (1-\alpha) \\ \beta & (1-\beta) \end{bmatrix}$$

Example 4.2 - A Communications System

A communications system transmits the digits 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability p that the digit entered will be unchanged when it leaves.

We let X_n denote the digit entering the nth stage.

Then $\{X_n : n \geq 0\}$ is a Markov chain with state space $\mathcal{S} = \{0, 1\}$, and with a transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{bmatrix} = \begin{bmatrix} p & (1-p) \\ (1-p) & p \end{bmatrix}$$

Example 4.3 - Mood swings

A person (Gary) has three mood states: cheerful, co-so and glum. The "mood" state space is denoted by $\mathcal{S} = \{0, 1, 2\}$, where 0 represents cheerful, 1 represents so-so and 2 represents glum.

We let X_n denote Gary's mood state at day n, $n = 0, 1, 2, \ldots$, and assume that $\{X_n\}$ is a Markov chain.

Thus, Gary's mood state tomorrow depends only on his mood state today, not on previous days.

The one-step transition probability matrix for $\{X_n\}$ is assumed to be:

$$\boldsymbol{P} = \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} \\ P_{1,0} & P_{1,1} & P_{1,2} \\ P_{2,0} & P_{2,1} & P_{2,2} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Example 4.3 - Mood swings (cont.)

$$\boldsymbol{P} = \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} \\ P_{1,0} & P_{1,1} & P_{1,2} \\ P_{2,0} & P_{2,1} & P_{2,2} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

NOTE 1: The highest numbers in this transition probability matrix are $P_{0,0}$ and $P_{2,2}$. Thus, if Gary is cheerful or glum one day, the most likely mood state the next day will be the same mood state.

NOTE 2: The lowest numbers in this transition probability matrix are $P_{2,0}$ and $P_{0,2}$. Thus, a change in mood state directly from cheerful to glum or the other way around are the least likely changes to occur.

Example 4.4 - Making a Process into a Markov chain

In this example consider a case where the weather conditions tomorrow depends on previous weather conditions through the last two days:

- If it rained today and yesterday, then the probability of rain tomorrow is 0.7.
- If it rained today but not yesterday, then the probability of rain tomorrow is 0.5.
- If it did not rain today but it rained yesterday, then the probability of rain tomorrow is 0.4.
- If it did not rain today and yesterday, then the probability of rain tomorrow is 0.2.

To make this into a Markov chain, we modify the state space so that each state describes the weather conditions in two consecutive days.

Making a Process into a Markov chain (cont.)

State space $\mathcal{S} = \{RR, \bar{R}R, R\bar{R}, \bar{R}\bar{R}\}$ where:

$$
\begin{aligned}
RR &= \text{Rain yesterday, Rain today,} \\
\bar{R}R &= \text{No rain yesterday, Rain today,} \\
R\bar{R} &= \text{Rain yesterday, No rain today,} \\
\bar{R}\bar{R} &= \text{No rain yesterday, No rain today.}
\end{aligned}
$$

NOTE: Assume that we e.g., are in state RR today, i.e., that it rained today and yesterday. Then for tomorrow's state, there are only two possibilities:

$$
\begin{array}{ll}
RR & \text{if it rains tomorrow} \\
R\bar{R} & \text{if it does not rain tomorrow}
\end{array}
$$

By the above assumptions, the probabilities of rain or not rain tomorrow given that we are in state state RR today, are:

$$P(RR|RR) = 0.70, \qquad P(R\bar{R}|RR) = 1 - 0.70 = 0.3$$

Making a Process into a Markov chain (cont.)

Following similar arguments, we get the following transition probabilities:

$$
\begin{aligned}
P\{RR|RR\} &= 0.70 & P\{R\bar{R}|RR\} &= 1 - 0.70 = 0.30 \\
P\{RR|\bar{R}R\} &= 0.50 & P\{R\bar{R}|\bar{R}R\} &= 1 - 0.50 = 0.50 \\
P\{\bar{R}R|R\bar{R}\} &= 0.40 & P\{\bar{R}\bar{R}|R\bar{R}\} &= 1 - 0.40 = 0.60 \\
P\{\bar{R}R|\bar{R}\bar{R}\} &= 0.20 & P\{\bar{R}\bar{R}|\bar{R}\bar{R}\} &= 1 - 0.20 = 0.80
\end{aligned}
$$

Thus, we have the following transition probability matrix:

$$
\boldsymbol{P} = \begin{bmatrix} 0.70 & 0.00 & 0.30 & 0.00 \\ 0.50 & 0.00 & 0.50 & 0.00 \\ 0.00 & 0.40 & 0.00 & 0.60 \\ 0.00 & 0.20 & 0.00 & 0.80 \end{bmatrix}
$$

where the row and column order is: RR, $\bar{R}R$, $R\bar{R}$, $\bar{R}\bar{R}$.

Example 4.5 - A Random Walk

A Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is said to be a random walk if, for some number $p \in (0, 1)$ we have:

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad \text{for all } i \in \mathcal{S}$$

A random walk is a stochastic model for the prosition of an individual walking on a straight line who at each point of time either takes one step to the right with probability p or one step to the left with probability $1 - p$.

NOTE: Assume that the initial state of the process, X_0 is zero. Then we have:

X_n is odd if n is odd

X_n is even if n is even

Example 4.6 - A Gambling Model

A random walk model with absorbing states is a Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, 2, \ldots, N\}$ where for some $p \in (0, 1)$:

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \quad i = 1, \ldots, N-1$$
$$P_{0,0} = P_{N,N} = 1$$

This process behaves like an ordinary random walk, except that the states 0 and N are absorbing. If the process reaches one of these states, the process stays in the same state indefinitely.

- X_n represents the gambler's fortune at time n
- In each game the gambler either wins one unit or loses one unit
- If $X_n = 0$, the gambler has spent his entire fortune, and cannot participate in the game.
- If $X_n = N$, the opponent has spent his entire fortune, and cannot participate in the game.

Example 4.7 - Automobile Insurance

The annual automobile insurance premium depends on the claim history of the client.

The client state in year n is modelled as a Markov chain $\{X_n\}$. This process can be in four different states $\mathcal{S} = \{1, 2, 3, 4\}$.

The state in a given year depends on the state in the previous year as well as the number of claims in the previous year.

State	**Annual Premium**	**Next state if**			
		0 claims	***1 claim***	***2 claims***	***$\geqslant$ 3 claims***
1	200	1	2	3	4
2	250	1	3	4	4
3	400	2	4	4	4
4	600	3	4	4	4

Figure: **Annual premium given number of claims**

Example 4.7 - Automobile Insurance (cont.)

Let Y_n denote the number of claims in year n. We assume that $Y_1, Y_2, \ldots$ are independent and identically distributed variables with:

$$P(Y_n = k) = a_k, \quad k = 0, 1, 2, \ldots$$

- If $X_n = 1$ and $Y_n = 0$, then $X_{n+1} = 1$.
  Thus, $P(X_{n+1} = 1 | X_n = 1) = a_0$
- If $X_n = 2$ and $Y_n = 0$, then $X_{n+1} = 1$.
  Thus, $P(X_{n+1} = 1 | X_n = 2) = a_0$
- If $X_n = 2$ and $Y_n = 1$, then $X_{n+1} = 3$.
  Thus, $P(X_{n+1} = 3 | X_n = 2) = a_1$
- $\ldots$

Example 4.7 - Automobile Insurance (cont.)

By similar arguments we find that the transition probability matrix of the process is:

$$\boldsymbol{P} = \begin{bmatrix} a_0 & a_1 & a_2 & 1 - a_0 - a_1 - a_2 \\ a_0 & 0 & a_1 & 1 - a_0 - a_1 \\ 0 & a_0 & 0 & 1 - a_0 \\ 0 & 0 & a_0 & 1 - a_0 \end{bmatrix}$$

Section 4.2 – Chapman-Kolmogorov Equations

Chapman-Kolmogorov Equations

***n*-step transition probabilities:**

$$P_{ij}^n = P\{X_{n+k} = j | X_k = i\}, \quad n \geq 1$$

***n*-step transition probability matrix:**

$$\boldsymbol{P}^{(n)} = [P_{ij}^n]_{i,j \in \mathcal{S}}$$

Chapman-Kolmogorov Equations: By conditioning on X_n we get:

$$P_{ij}^{n+m} = \sum_{k \in \mathcal{S}} P_{ik}^n \cdot P_{kj}^m$$

By using the definition of matrix products it follows that:

$$\boldsymbol{P}^{(n+m)} = \boldsymbol{P}^{(n)} \cdot \boldsymbol{P}^{(m)}$$

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^{(n-1)} \cdot \boldsymbol{P}^{(1)}$$

$$\boldsymbol{P}^{(n)} = \boldsymbol{P}^{n}$$

Example 4.8

$$P\{\text{Rain tomorrow}|\text{Rain today}\} = 0.75$$
$$P\{\text{Rain tomorrow}|\text{No rain today}\} = 0.35$$

$$\boldsymbol{P} = \begin{bmatrix} 0.75 & 0.25 \\ 0.35 & 0.65 \end{bmatrix}$$

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} 0.65 & 0.35 \\ 0.49 & 0.51 \end{bmatrix}$$

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.5940 & 0.4060 \\ 0.5684 & 0.4316 \end{bmatrix}$$

$$\boldsymbol{P}^{(8)} = \begin{bmatrix} 0.5836 & 0.4164 \\ 0.5830 & 0.4170 \end{bmatrix}$$

Example 4.9

$$
\begin{aligned}
RR &= \text{Rain yesterday, Rain today,} \\
\bar{R}R &= \text{No rain yesterday, Rain today,} \\
R\bar{R} &= \text{Rain yesterday, No rain today,} \\
\bar{R}\bar{R} &= \text{No rain yesterday, No rain today.}
\end{aligned}
$$

$$
\begin{aligned}
P\{RR|RR\} &= 0.70 & P\{R\bar{R}|RR\} &= 0.30 \\
P\{RR|\bar{R}R\} &= 0.50 & P\{R\bar{R}|\bar{R}R\} &= 0.50 \\
P\{\bar{R}R|R\bar{R}\} &= 0.40 & P\{\bar{R}\bar{R}|R\bar{R}\} &= 0.60 \\
P\{\bar{R}R|\bar{R}\bar{R}\} &= 0.20 & P\{\bar{R}\bar{R}|\bar{R}\bar{R}\} &= 0.80
\end{aligned}
$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $R\bar{R}$, $\bar{R}\bar{R}$.

$$\boldsymbol{P} = \begin{bmatrix} 0.70 & 0.00 & 0.30 & 0.00 \\ 0.50 & 0.00 & 0.50 & 0.00 \\ 0.00 & 0.40 & 0.00 & 0.60 \\ 0.00 & 0.20 & 0.00 & 0.80 \end{bmatrix}$$

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} \color{red}{0.49} & \color{red}{0.12} & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix}$$

$$P\{\text{Rain Thursday} | \text{Rain Monday \& Rain Tuesday}\} = 0.49 + 0.12 = 0.61$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $R\bar{R}$, $\bar{R}\bar{R}$.

$$\boldsymbol{P}^{(7)} = \begin{bmatrix} \color{red}{0.2723} & \color{red}{0.1465} & 0.1580 & 0.4233 \\ 0.2633 & 0.1477 & 0.1549 & 0.4340 \\ 0.2441 & 0.1511 & 0.1477 & 0.4571 \\ 0.2352 & 0.1524 & 0.1447 & 0.4678 \end{bmatrix}$$

$$P\{\text{Rain next Tuesday}|\text{Rain Monday \& Rain Tuesday}\}$$

$$= 0.2723 + 0.1465 = 0.4188$$

Example 4.9 (cont.)

Row/Column order: RR, $\bar{R}R$, $R\bar{R}$, $\bar{R}\bar{R}$.

$$\boldsymbol{P}^{(20)} \approx \begin{bmatrix} 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \\ 0.25 & 0.15 & 0.15 & 0.50 \end{bmatrix}$$

$$P\{\text{Rain "some day" in the future}\}$$

$$\approx P\{\text{Rain "some day" in the future}|\text{Rain Monday \& Rain Tuesday}\}$$

$$\approx 0.25 + 0.15 = 0.40$$

Example 4.10

An urn always contains 2 balls. Ball colors are red and blue.

At each stage a ball is randomly chosen and then replaced by a new ball, which with probability 0.8 is the *same color*, and with probability 0.2 is the *opposite color*, as the ball it replaces.

If initially both balls are red, find the probability that the fifth ball selected is red.

$$X_n = \text{The number of red balls after the } n\text{th experiment.} \quad n = 0, 1, 2, \ldots$$

Example 4.10 (cont.)

NOTE: $X_n \in \{0, 1, 2\}$, for all $n = 0, 1, 2, \dots$

We then have:

$$P_{00} = 0.8, \qquad P_{01} = 0.2, \qquad P_{02} = 0.0$$

$$\begin{aligned} P_{10} &= P\{\text{Red ball selected and replaced}\} = 0.5 \cdot 0.2 = 0.1 \\ P_{11} &= P\{\text{Any ball selected and not replaced}\} = 0.8 \\ P_{12} &= P\{\text{Blue ball selected and replaced}\} = 0.5 \cdot 0.2 = 0.1 \end{aligned}$$

$$P_{20} = 0.0, \qquad P_{21} = 0.2, \qquad P_{22} = 0.8$$

$$\boldsymbol{P} = \begin{bmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.8 & 0.1 \\ 0.0 & 0.2 & 0.8 \end{bmatrix}$$

Example 4.10 (cont.)

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{bmatrix}$$

P(Selection 5 is red)

$$= \sum_{i=0}^{2} P(\text{Selection 5 is red}|X_4 = i) \cdot P(X_4 = i|X_0 = 2)$$

$$= 0.00 \cdot P^4_{2,0} + 0.50 \cdot P^4_{2,1} + 1.00 \cdot P^4_{2,2}$$

$$= 0.50 \cdot 0.4352 + 0.4872 = 0.7048$$

Example 4.10 (cont.)

If we instead started out with one blue and one red ball, i.e., if $X_0 = 1$, we would get the following:

$$\boldsymbol{P}^{(4)} = \begin{bmatrix} 0.4872 & 0.4352 & 0.0776 \\ 0.2176 & 0.5648 & 0.2176 \\ 0.0776 & 0.4352 & 0.4872 \end{bmatrix}$$

$$P(\text{Selection 5 is red})$$

$$= \sum_{i=0}^{2} P(\text{Selection 5 is red} | X_4 = i) \cdot P(X_4 = i | X_0 = 1)$$

$$= 0.00 \cdot P^4_{1,0} + 0.50 \cdot P^4_{1,1} + 1.00 \cdot P^4_{1,2}$$

$$= 0.50 \cdot 0.5648 + 0.2176 = 0.5$$

Example 4.10 (cont.)

$$\boldsymbol{P}^{(30)} \approx \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.50 & 0.25 \end{bmatrix}$$

$P(\text{Selection 31 is red})$

$$= \sum_{i=0}^{2} P(\text{Selection 31 is red} | X_{30} = i) \cdot P(X_{30} = i | X_0 = 2)$$

$$= 0.00 \cdot P_{2,0}^{30} + 0.50 \cdot P_{2,1}^{30} + 1.00 \cdot P_{2,2}^{30}$$

$$\approx 0.50 \cdot 0.50 + 0.25 = 0.50$$

Example 4.11

Suppose that balls are successively distributed among 8 urns, with each ball being equally likely to be put in any of these urns.

PROBLEM: What is the probability that there will be exactly 3 nonempty urns after 9 balls have been distributed?

To solve this problem we introduce a Markov chain $\{X_n\}$, where:

$$X_n = \text{Number of nonempty urns after } n \text{ distributions} \quad n = 0, 1, 2 \dots$$

In particular, we have that $X_0 = 0$.

Moreover, we have the following transition probabilities:

$$P_{i,i} = \frac{i}{8}, \qquad P_{i,i+1} = \frac{8-i}{8}, \quad i = 0, 1, \dots, 8.$$

Example 4.11 (cont.)

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/8 & 7/8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/8 & 6/8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/8 & 5/8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/8 & 4/8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6/8 & 2/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7/8 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By calculating $\boldsymbol{P}^{(9)}$ we find the solution to the problem:

$$P(X_9 = 3 | X_0 = 0) = P_{03}^9 = 0.00756.$$

However, there are some significant shortcuts here.

Example 4.11 (cont.)

The first transition of $\{X_n\}$ is deterministic (from state 0 to 1). Thus, we may instead consider the Markov chain $\{Y_n\}$ instead:

$$Y_n = X_{n+1}, \quad n = 0, 1, 2, \ldots,$$

with state space $\{1, 2, \ldots, 8\}$ and use that:

$$P(X_9 = 3 | X_0 = 0) = P(Y_8 = 3 | Y_0 = 1)$$

We can simplify the problem even further by letting:

$$Z_n = \min\{Y_n, 4\}, \quad n = 0, 1, 2, \ldots,$$

and use that:

$$P(X_9 = 3 | X_0 = 0) \ = \ P(Y_8 = 3 | Y_0 = 1) \ = \ P(Z_8 = 3 | Z_0 = 1)$$

Example 4.11 (cont.)

The state space of the Markov chain $\{Z_n\}$ is $\{1, 2, 3, 4\}$ and its transition matrix is:

$$\boldsymbol{Q} = \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.11 (cont.)

We first calculate:

$$\boldsymbol{Q}^{(4)} = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 & 0.7178 \\ 0.0000 & 0.0039 & 0.0952 & 0.9009 \\ 0.0000 & 0.0000 & 0.0198 & 0.9802 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

In order to find $\boldsymbol{Q}^{(8)}$, we may use that:

$$\boldsymbol{Q}^{(8)} = \boldsymbol{Q}^{(4)} \cdot \boldsymbol{Q}^{(4)}$$

However, since we only need to find $P(X_9 = 3 | X_0 = 0)$, we only need to calculate:

$$P(X_9 = 3|X_0 = 0) = P(Z_8 = 3|Z_0 = 1) = Q^8_{1,3} = \sum_{k=1}^{4} Q^4_{1,k} \cdot Q^4_{k,3}$$

$$= 0.0002 \cdot 0.2563 + 0.0256 \cdot 0.0952 + 0.2563 \cdot 0.0198 = 0.00756.$$