STK2130 – Lecture 13

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A continuous-time Markov chain

We recall the following from Chapter 6.2:

A continuous-time Markov chain with stationary transition probabilities and state space $\mathcal{X}$ is a stochastic process such that:

- The times spent in the different states are independent random variables (because of the Markov property).
- The amount of time spent in state $i \in \mathcal{X}$ is exponentially distributed with rate v_i (because of the Markov property and stationarity).
- When the process leaves state i, it enters state j with some transition probability Q_{ij} where:

$$Q_{ii} = 0, \quad \text{for all } i \in \mathcal{X}$$

$$\sum_{j \in \mathcal{X}} Q_{ij} = 1, \quad \text{for all } i \in \mathcal{X}$$

- The transitions follow a discrete-time Markov chain.

A continuous-time Markov chain (cont.)

We now introduce the following notation:

$$q_{ij} = v_i Q_{ij}, \quad i, j \in \mathcal{X}.$$

A continuous-time Markov chain (cont.)

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INTERPRETATION: Since v_i is the rate at which the process makes a transition when in state i and Q_{ij} is the probability that this transition is into state j, it follows that q_{ij} is the rate, when in state i, at which the process makes a transition into state j.

The quantities q_{ij} are called the instantaneous transition rates.

A continuous-time Markov chain (cont.)

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The quantities q_{ij} are called the instantaneous transition rates.

Since we have:

$$v_i = v_i \sum_{j \in \mathcal{X}} Q_{ij} = \sum_{j \in \mathcal{X}} v_i Q_{ij} = \sum_{j \in \mathcal{X}} q_{ij},$$

$$Q_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_{j \in \mathcal{X}} q_{ij}},$$

the probabilistic properties of $\{X(t) : t \geq 0\}$ is determined by the q_{ij}'s.

6.4 Kolmogorov's Backward Equations

Lemma (6.2)

Let $P_{ij}(t) = P(X(t) = j | X(0) = i)$*,* $i, j \in \mathcal{X}$*. We then have:*

$$(a)\ \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i$$

$$(b)\ \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}, \quad \text{for all } i \neq j$$

6.4 Kolmogorov's Backward Equations

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PROOF: We start by noting that the amount of time until a transition occurs is exponentially distributed.

6.4 Kolmogorov's Backward Equations

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PROOF: We start by noting that the amount of time until a transition occurs is exponentially distributed.

Hence, the probability of two or more transitions in a time h is $o(h)$.

6.4 Kolmogorov's Backward Equations (cont.)

Hence, it follows that:

$$P(X(h) \neq i | X(0) = i) = 1 - P(X(h) = i | X(0) = i) = 1 - P_{ii}(h) = v_i h + o(h).$$

6.4 Kolmogorov's Backward Equations (cont.)

Hence, it follows that:

$$P(X(h) \neq i | X(0) = i) = 1 - P(X(h) = i | X(0) = i) = 1 - P_{ii}(h) = v_i h + o(h).$$

By dividing both sides by h and letting h go to 0, we get:

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h \to 0} \frac{v_i h + o(h)}{h} = v_i + \lim_{h \to 0} \frac{o(h)}{h} = v_i$$

which proves (a).

6.4 Kolmogorov's Backward Equations (cont.)

Similarly, if $i \neq j$, we get:

$$P(X(h) = j | X(0) = i) = P_{ij}(h) = v_i Q_{ij} h + o(h).$$

6.4 Kolmogorov's Backward Equations (cont.)

Similarly, if $i \neq j$, we get:

$$P(X(h) = j | X(0) = i) = P_{ij}(h) = v_i Q_{ij} h + o(h).$$

By dividing both sides by h and letting h go to 0, we get:

$$\lim_{h \to 0} \frac{P_{ij}(h)}{h} = \lim_{h \to 0} \frac{v_i Q_{ij} h + o(h)}{h} = v_i Q_{ij} + \lim_{h \to 0} \frac{o(h)}{h} = v_i Q_{ij} = q_{ij}$$

which proves (b).

6.4 Kolmogorov's Backward Equations (cont.)

Lemma (6.3 – Chapman-Kolmogorov equations)

For all $s, t \geq 0$ *and* $i, j \in \mathcal{X}$ *we have:*

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s)$$

6.4 Kolmogorov's Backward Equations (cont.)

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For all $s, t \geq 0$ and $i, j \in \mathcal{X}$ we have:

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s)$$

PROOF: We have:

$$\begin{aligned} P_{ij}(t+s) &= P(X(t+s) = j | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j, X(t) = k | X(0) = i) \end{aligned}$$

6.4 Kolmogorov's Backward Equations (cont.)

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PROOF: We have:

$$\begin{aligned} P_{ij}(t+s) &= P(X(t+s) = j | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j, X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j | X(t) = k, X(0) = i) \cdot P(X(t) = k | X(0) = i) \end{aligned}$$

6.4 Kolmogorov's Backward Equations (cont.)

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P(X(t+s) = j | X(t) = k, X(0) = i) \cdot P(X(t) = k | X(0) = i)$$

6.4 Kolmogorov's Backward Equations (cont.)

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&= \sum_{k \in \mathcal{X}} P(X(s) = j | X(0) = k) \cdot P(X(t) = k | X(0) = i) \\
&= \sum_{k \in \mathcal{X}} P_{kj}(s) \cdot P_{ik}(t)
\end{aligned}
$$

$\blacksquare$

6.4 Kolmogorov's Backward Equations (cont.)

Theorem (6.1 – Kolmogorov's backward equations)

For all $t \geq 0$ and states $i, j \in \mathcal{X}$ we have:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

6.4 Kolmogorov's Backward Equations (cont.)

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PROOF: By Lemma 6.3 we have:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

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PROOF: By Lemma 6.3 we have:

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t) - P_{ij}(t) \\ &= \sum_{k \in \mathcal{X} \setminus i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t) - P_{ij}(t) \end{aligned}$$

6.4 Kolmogorov's Backward Equations (cont.)

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For all $t \geq 0$ and states $i, j \in \mathcal{X}$ we have:

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PROOF: By Lemma 6.3 we have:

$$\begin{aligned}
P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t) - P_{ij}(t) \\
&= \sum_{k \in \mathcal{X} \setminus i} P_{ik}(h) P_{kj}(t) + P_{ii}(h) P_{ij}(t) - P_{ij}(t) \\
&= \sum_{k \in \mathcal{X} \setminus i} P_{ik}(h) P_{kj}(t) - [1 - P_{ii}(h)] P_{ij}(t)
\end{aligned}$$

6.4 Kolmogorov's Backward Equations (cont.)

By dividing both sides by h and letting $h \to 0$, we can use Lemma 6.2 and get:

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \in \mathcal{X} \setminus i} \lim_{h \to 0} \frac{P_{ik}(h)}{h} P_{kj}(t) - \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

$$= \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

■

6.4 Kolmogorov's Backward Equations (cont.)

Kolmogorov's backward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t), \quad i, j \in \mathcal{X}.$$

We recall that $v_i = \sum_{k \in \mathcal{X}} q_{i,k} = \sum_{k \in \mathcal{X}} v_i Q_{i,k}$. Since $Q_{i,i} = 0$, this implies that:

$$v_i = \sum_{k \in \mathcal{X} \setminus i} q_{i,k}$$

Hence, we have:

$$\sum_{k \in \mathcal{X} \setminus i} q_{i,k} - v_i = \sum_{k \in \mathcal{X} \setminus i} q_{i,k} - \sum_{k \in \mathcal{X} \setminus i} q_{i,k} = 0$$

Thus, in each of the backward equations the sum of coefficients is equal to zero.

6.4 Kolmogorov's Backward Equations (cont.)

We now assume that $\mathcal{X} = \{1, 2, \ldots, n\}$, and introduce the following matrices:

$$\boldsymbol{R} = \begin{bmatrix} -v_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ q_{2,1} & -v_2 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & -v_3 & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & -v_n \end{bmatrix}$$

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{1,1}(t) & P_{1,2}(t) & P_{1,3}(t) & \cdots & P_{1,n}(t) \\ P_{2,1}(t) & P_{2,2}(t) & P_{2,3}(t) & \cdots & P_{2,n}(t) \\ P_{3,1}(t) & P_{3,2}(t) & P_{3,3}(t) & \cdots & P_{3,n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n,1}(t) & P_{n,2}(t) & P_{n,3}(t) & \cdots & P_{n,n}(t) \end{bmatrix}$$

6.4 Kolmogorov's Backward Equations (cont.)

It is then easy to see that Kolmogorov's backward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

can be written in the following form:

$$\boldsymbol{P}'(t) = \boldsymbol{R}\boldsymbol{P}(t).$$

Example 6.10 – Pure birth process

For a pure birth process with birth rates $\lambda_0, \lambda_1, \ldots$ we have:

$$q_{i,i+1} = \lambda_i, \quad i = 0, 1, \ldots$$

$$q_{i,j} = 0, \quad \text{for all } j \neq (i+1)$$

Hence, we also have:

$$v_i = \sum_{j=0}^{\infty} q_{ij} = q_{i,i+1} = \lambda_i, \quad i = 0, 1, \ldots$$

By inserting this into the backward equations we get:

$$\begin{aligned} P'_{ij}(t) &= \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \\ &= \lambda_i P_{i+1,j}(t) - \lambda_i P_{ij}(t), \quad t \geq 0, \quad 0 \leq i \leq j \end{aligned}$$

Example 6.10 – Birth and death process

For a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_1, \mu_2, \ldots$, we have:

$$
\begin{aligned}
q_{i,i+1} &= \lambda_i, \quad i = 0, 1, \ldots \\
q_{i,i-1} &= \mu_i, \quad i = 1, 2, \ldots \\
q_{i,j} &= 0, \quad \text{otherwise}
\end{aligned}
$$

Hence, we also have:

$$v_0 = \sum_{j=0}^{\infty} q_{0j} = \lambda_0$$

$$v_i = \sum_{j=0}^{\infty} q_{ij} = \lambda_i + \mu_i, \quad i = 1, 2, \ldots$$

Example 6.10 – Birth and death process (cont.)

By inserting this into the backward equations we get:

$$
\begin{aligned}
P'_{0j}(t) &= \sum_{k \in \mathcal{X} \setminus 0} q_{0k} P_{kj}(t) - v_0 P_{0j}(t) \\
&= \lambda_0 P_{1,j}(t) - \lambda_0 P_{0j}(t), \quad t \geq 0, \quad j \geq 0
\end{aligned}
$$

$$
\begin{aligned}
P'_{ij}(t) &= \sum_{k \in \mathcal{X} \setminus i} q_{ik} P_{kj}(t) - v_i P_{ij}(t) \\
&= \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t), \quad t \geq 0, \quad i > 0, j \geq 0
\end{aligned}
$$

Example 6.11

The lifetimes and repair times of a system are independent and exponentially distributed with rates respectively λ and μ.

We model this system as a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with state space $\mathcal{X} = \{0, 1\}$, where[1]:

$$X(t) = I(\text{The system is functioning at time } t), \quad t \geq 0.$$

The only non-zero instantaneous transition rates are $q_{01} = \mu$ and $q_{10} = \lambda$. Hence;

$$v_0 = \sum_{j \in \mathcal{X}} q_{0j} = q_{01} = \mu$$

$$v_1 = \sum_{j \in \mathcal{X}} q_{1j} = q_{10} = \lambda$$

[1] In the Ross(2019) state 0 is the functioning state and state 1 is the failed state.

Example 6.11 (cont.)

By inserting this into the backward equations we get:

$$P'_{00}(t) = \mu P_{10}(t) - \mu P_{00}(t) = \mu[P_{10}(t) - P_{00}(t)], \tag{1}$$

$$P'_{01}(t) = \mu P_{11}(t) - \mu P_{01}(t) = \mu[P_{11}(t) - P_{01}(t)], \tag{2}$$

$$P'_{10}(t) = \lambda P_{00}(t) - \lambda P_{10}(t) = \lambda[P_{00}(t) - P_{10}(t)], \tag{3}$$

$$P'_{11}(t) = \lambda P_{01}(t) - \lambda P_{11}(t) = \lambda[P_{01}(t) - P_{11}(t)] \tag{4}$$

We start by computing $P_{11}(t)$. In order to solve this, we multiply (2) by λ and multiply (4) by μ and add the resulting equations:

$$\mu P'_{11}(t) + \lambda P'_{01}(t) = 0$$

By integrating both sides we get:

$$\mu P_{11}(t) + \lambda P_{01}(t) = c$$

Example 6.11 (cont.)

In order to determine the constant c, we note that $P_{11}(0) = 1$, while $P_{01}(0) = 0$. Hence, $c = \mu$, and we get:

$$\mu P_{11}(t) + \lambda P_{01}(t) = \mu$$

or equivalently:

$$\lambda P_{01}(t) = \mu[1 - P_{11}(t)]$$

We insert this into the right-hand side of (4) $P'_{11}(t) = \lambda P_{01}(t) - \lambda P_{11}(t)$ and get:

$$\begin{aligned} P'_{11}(t) &= \mu[1 - P_{11}(t)] - \lambda P_{11}(t) \\ &= \mu - (\mu + \lambda) P_{11}(t) \end{aligned}$$

or equivalently:

$$P'_{11}(t) + (\mu + \lambda) P_{11}(t) = \mu$$

Example 6.11 (cont.)

In order to solve the differential equation $P'_{11}(t) + (\mu + \lambda)P_{11}(t) = \mu$, we multiply both sides by the integrating factor $e^{(\mu+\lambda)t}$ and get:

$$P'_{11}(t)e^{(\mu+\lambda)t} + (\mu + \lambda)e^{(\mu+\lambda)t}P_{11}(t) = \mu e^{(\mu+\lambda)t}$$

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In order to solve the differential equation $P'_{11}(t) + (\mu + \lambda)P_{11}(t) = \mu$, we multiply both sides by the integrating factor $e^{(\mu+\lambda)t}$ and get:

$$P'_{11}(t)e^{(\mu+\lambda)t} + (\mu + \lambda)e^{(\mu+\lambda)t}P_{11}(t) = \mu e^{(\mu+\lambda)t}$$

By using the product rule for derivatives, the left-hand side can be simplified to:

$$(P_{11}(t)e^{(\mu+\lambda)t})' = \mu e^{(\mu+\lambda)t}$$

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By integrating both sides of this equation we get:

$$P_{11}(t)e^{(\mu+\lambda)t} = \frac{\mu}{\mu + \lambda}e^{(\mu+\lambda)t} + C.$$

Example 6.11 (cont.)

In order to solve the differential equation $P'_{11}(t) + (\mu + \lambda)P_{11}(t) = \mu$, we multiply both sides by the integrating factor $e^{(\mu+\lambda)t}$ and get:

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By integrating both sides of this equation we get:

$$P_{11}(t)e^{(\mu+\lambda)t} = \frac{\mu}{\mu + \lambda}e^{(\mu+\lambda)t} + C.$$

or equivalently:

$$P_{11}(t) = \frac{\mu}{\mu + \lambda} + Ce^{-(\mu+\lambda)t}$$

Example 6.11 (cont.)

In order to determine the constant C, we again use that $P_{11}(0) = 1$. That is:

$$1 = \frac{\mu}{\mu + \lambda} + C$$

Hence, C is given by:

$$C = 1 - \frac{\mu}{\mu + \lambda} = \frac{\lambda}{\mu + \lambda},$$

and thus:

$$P_{11}(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\mu+\lambda)t}$$

Example 6.11 (cont.)

This also implies that:

$$
\begin{aligned}
P_{10}(t) &= 1 - P_{11}(t) \\
&= 1 - \frac{\mu}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} \\
&= \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}
\end{aligned}
$$

We recall that we also have established that $\mu P_{11}(t) + \lambda P_{01}(t) = \mu$. Hence, we get that:

$$
\begin{aligned}
P_{01}(t) &= \frac{\mu}{\lambda}[1 - P_{11}(t)] = \frac{\mu}{\lambda} P_{10}(t) \\
&= \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}
\end{aligned}
$$

Example 6.11 (cont.)

Finally, we get that:

$$
\begin{aligned}
P_{00}(t) &= 1 - P_{01}(t) \\
&= 1 - \frac{\mu}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t} \\
&= \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}
\end{aligned}
$$

Example 6.11 (cont.)

Summarizing all these results we get:

$$P_{11}(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}$$

$$P_{10}(t) = \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}$$

$$P_{01}(t) = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}$$

$$P_{00}(t) = \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}$$

Example 6.11 (cont.)

We observe that:

$$\lim_{t \to \infty} P_{11}(t) = \lim_{t \to \infty} P_{01}(t) = \frac{\mu}{\mu + \lambda} = \frac{\lambda^{-1}}{\mu^{-1} + \lambda^{-1}}.$$

and that:

$$\lim_{t \to \infty} P_{00}(t) = \lim_{t \to \infty} P_{10}(t) = \frac{\lambda}{\mu + \lambda} = \frac{\mu^{-1}}{\mu^{-1} + \lambda^{-1}}.$$

6.4 Kolmogorov's Forward Equations

Theorem (6.2 – Kolmogorov's forward equations)

For all $t \geq 0$ and states $i, j \in \mathcal{X}$ we have:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

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PROOF: By Lemma 6.3 we have:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

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PROOF: By Lemma 6.3 we have:

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(h) - P_{ij}(t) \\ &= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) P_{kj}(h) + P_{ij}(t) P_{jj}(h) - P_{ij}(t) \end{aligned}$$

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PROOF: By Lemma 6.3 we have:

$$\begin{aligned}
P_{ij}(t+h) - P_{ij}(t) &= \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(h) - P_{ij}(t) \\
&= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) P_{kj}(h) + P_{ij}(t) P_{jj}(h) - P_{ij}(t) \\
&= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) P_{kj}(h) - P_{ij}(t)[1 - P_{jj}(h)]
\end{aligned}$$

6.4 Kolmogorov's Forward Equations (cont.)

By dividing both sides by h and letting $h \to 0$, we can use Lemma 6.2[2] and get:

$$\lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) \lim_{h \to 0} \frac{P_{kj}(h)}{h} - P_{ij}(t) \lim_{h \to 0} \frac{1 - P_{jj}(h)}{h}$$

$$= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$$

■

[2] Unfortunately, the interchange of limit and summation is not always valid. This holds, however, for all birth and death processes and for all finite state models

6.4 Kolmogorov's Forward Equations (cont.)

We again assume that $\mathcal{X} = \{1, 2, \ldots, n\}$, and recall the following matrices:

$$\boldsymbol{R} = \begin{bmatrix} -v_1 & q_{1,2} & q_{1,3} & \cdots & q_{1,n} \\ q_{2,1} & -v_2 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & -v_3 & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & -v_n \end{bmatrix}$$

$$\boldsymbol{P}(t) = \begin{bmatrix} P_{1,1}(t) & P_{1,2}(t) & P_{1,1}(t) & \cdots & P_{1,n}(t) \\ P_{2,1}(t) & P_{2,2}(t) & P_{2,3}(t) & \cdots & P_{2,n}(t) \\ P_{3,1}(t) & P_{3,2}(t) & P_{3,3}(t) & \cdots & P_{3,n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{n,1}(t) & P_{n,2}(t) & P_{n,3}(t) & \cdots & P_{n,n}(t) \end{bmatrix}$$

6.4 Kolmogorov's Forward Equations (cont.)

It is then easy to see that Kolmogorov's forward equations:

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j.$$

can be written in the following form:

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R}.$$

Pure birth processes

Assume that $\{X(t) : t \geq 0\}$ is a pure birth process with birth rates $\lambda_0, \lambda_1, \ldots$.
From this it follows that we have:

$$q_{i,i+1} = \lambda_i, \quad i = 0, 1, \ldots$$

$$q_{i,j} = 0, \quad \text{for all } j \neq (i+1)$$

Hence, we also have:

$$v_i = \sum_{j=0}^{\infty} q_{ij} = q_{i,i+1} = \lambda_i, \quad i = 0, 1, \ldots$$

By inserting this into the forward equations we get:

$$\begin{aligned} P'_{ij}(t) &= \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j \\ &= P_{i,j-1}(t)\lambda_{j-1} - P_{ij}(t)\lambda_j, \quad t \geq 0, \quad i, j = 0, 1, \ldots \end{aligned}$$

Pure birth processes (cont.)

Note, however, that $P_{ij}(t) = 0$ whenever $j < i$ (since this is a pure birth process with no deaths).

Hence, we get the following simplified equations:

$$P'_{ii}(t) = \lambda_{i-1} P_{i,i-1}(t) - \lambda_i P_{ii}(t) = -\lambda_i P_{ii}(t), \quad i = 0, 1, \ldots$$

$$P'_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{ij}(t), \quad j \geq i + 1$$

Pure birth processes (cont.)

Proposition (6.4 – Pure birth processes)

Assume that $\{X(t) : t \geq 0\}$ *is a pure birth process with birth rates* $\lambda_0, \lambda_1, \ldots$. *We then have:*

$$P_{ii}(t) = e^{-\lambda_i t}, \quad i = 0, 1, 2, \ldots$$

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds, \quad j \geq i + 1.$$

Pure birth processes (cont.)

We start out by noting that since $P'_{ii}(t) = -\lambda_i P_{ii}(t)$, it follows that

$$\frac{P'_{ii}(t)}{P_{ii}(t)} = [\ln(P_{ii}(t))]' = -\lambda_i$$

By integrating both sides, we get:

$$\ln(P_{ii}(t)) = -\lambda_i t + c$$

This implies that:

$$P_{ii}(t) = e^{-\lambda_i t + c}$$

Since $P_{ii}(0) = 1$, it follows that $c = 0$. Hence, we get:

$$P_{ii}(t) = e^{-\lambda_i t}$$

Pure birth processes (cont.)

In order to prove the corresponding result for $P_{ij}(t)$ where $j \geq i+1$, we assume that we have determined $P_{i,j-1}(t)$ already, and rewrite the differential equation as:

$$P'_{ij}(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t)$$

We then multiply both sides by the integrating factor $e^{\lambda_j t}$ and get:

$$P'_{ij}(t) e^{\lambda_j t} + \lambda_j e^{\lambda_j t} P_{ij}(t) = \lambda_{j-1} e^{\lambda_j t} P_{i,j-1}(t).$$

Pure birth processes (cont.)

In order to prove the corresponding result for $P_{ij}(t)$ where $j \geq i+1$, we assume that we have determined $P_{i,j-1}(t)$ already, and rewrite the differential equation as:

$$P_{ij}'(t) + \lambda_j P_{ij}(t) = \lambda_{j-1} P_{i,j-1}(t)$$

We then multiply both sides by the integrating factor $e^{\lambda_j t}$ and get:

$$P_{ij}'(t) e^{\lambda_j t} + \lambda_j e^{\lambda_j t} P_{ij}(t) = \lambda_{j-1} e^{\lambda_j t} P_{i,j-1}(t).$$

Using the product rule for derivatives, the left-hand side can be simplified to:

$$(P_{ij}(t) e^{\lambda_j t})' = \lambda_{j-1} e^{\lambda_j t} P_{i,j-1}(t)$$

Pure birth processes (cont.)

In order to prove the corresponding result for $P_{ij}(t)$ where $j \geq i+1$, we assume that we have determined $P_{i,j-1}(t)$ already, and rewrite the differential equation as:

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Using the product rule for derivatives, the left-hand side can be simplified to:

$$(P_{ij}(t) e^{\lambda_j t})' = \lambda_{j-1} e^{\lambda_j t} P_{i,j-1}(t)$$

Integrating both sides of this equation we get:

$$P_{ij}(t) e^{\lambda_j t} = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds + C$$

Pure birth processes (cont.)

We note that:

$$P_{ij}(t)e^{\lambda_j t} = \lambda_{j-1} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds + C$$

is equivalent to:

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds + Ce^{-\lambda_j t}$$

In order to determine the constant C we use that when $j \geq i + 1$, we have $P_{ij}(0) = 0$. Hence, we must have $C = 0$, and we conclude that:

$$P_{ij}(t) = \lambda_{j-1} e^{-\lambda_j t} \int_0^t e^{\lambda_j s} P_{i,j-1}(s) ds$$

■

Example 6.12 – Birth and death processes

We recall that for a birth and death process with birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_1, \mu_2, \ldots$ we have:

$$
\begin{aligned}
q_{i,i+1} &= \lambda_i, \quad i = 0, 1, \ldots \\
q_{i,i-1} &= \mu_i, \quad i = 1, 2, \ldots \\
q_{i,j} &= 0, \quad \text{otherwise}
\end{aligned}
$$

Example 6.12 – Birth and death processes

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q_{i,i+1} &= \lambda_i, \quad i = 0, 1, \ldots \\
q_{i,i-1} &= \mu_i, \quad i = 1, 2, \ldots \\
q_{i,j} &= 0, \quad \text{otherwise}
\end{aligned}
$$

Hence, we also have:

$$
\begin{aligned}
v_0 &= \sum_{j=0}^{\infty} q_{0j} = \lambda_0 \\
v_i &= \sum_{j=0}^{\infty} q_{ij} = \lambda_i + \mu_i, \quad i = 1, 2, \ldots
\end{aligned}
$$

Example 6.12 (cont.)

We insert this into the forward equations, handling the case where $j = 0$ separately:

$$
\begin{aligned}
P'_{i0}(t) &= \sum_{k \in \mathcal{X} \setminus 0} P_{ik}(t) q_{k0} - P_{i0}(t) v_0 \\
&= P_{i1}(t) q_{10} - P_{i0}(t) v_0 = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)
\end{aligned}
$$

Example 6.12 (cont.)

We insert this into the forward equations, handling the case where $j = 0$ separately:

$$P'_{i0}(t) = \sum_{k \in \mathcal{X} \setminus 0} P_{ik}(t) q_{k0} - P_{i0}(t) v_0$$

$$= P_{i1}(t) q_{10} - P_{i0}(t) v_0 = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t)$$

$$P'_{ij}(t) = \sum_{k \in \mathcal{X} \setminus j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$$

$$= P_{i,j-1}(t) q_{j-1,j} + P_{i,j+1}(t) q_{j+1,j} - P_{ij}(t) v_j$$

$$= \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t)$$