STK2130 – Lecture 3

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Section 4.3 – Classification of States

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Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S}$ and transition probability matrix $\boldsymbol{P}$.

State j is said to be accessible from state i, denoted as $i \to j$, if $P_{ij}^n > 0$ for some $n \geq 0$.

Note that we have:

$$P_{ij}^n \leq P(\bigcup_{n=0}^{\infty}\{X_n = j\}|X_0 = i), \quad n = 0, 1, 2, \ldots$$

Hence, we also have:

$$\sup_{n \geq 0} P_{ij}^n \leq P(\bigcup_{n=0}^{\infty}\{X_n = j\}|X_0 = i)$$
$$\leq \sum_{n=0}^{\infty} P\{X_n = j|X_0 = i\} = \sum_{n=0}^{\infty} P_{ij}^n.$$

If $i \to j$, then by definition $P_{ij}^n > 0$ for some $n \geq 0$, and hence we obviously also have that $\sup_n P_{ij}^n > 0$. This implies that:

$$P(\bigcup_{n=0}^{\infty} \{X_n = j\} | X_0 = i) \geq \sup_{n \geq 0} P_{ij}^n > 0.$$

Conversely, if $P(\bigcup_{n=0}^{\infty} \{X_n = j\} | X_0 = i) > 0$, then also:

$$\sum_{n=0}^{\infty} P_{ij}^n \geq P(\bigcup_{n=0}^{\infty} \{X_n = j\} | X_0 = i) > 0$$

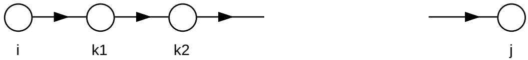
which implies that $P_{ij}^n > 0$ for some $n \geq 0$, i.e., that $i \to j$.

Hence, we conclude that $i \to j$ if and only if:

$$P(\bigcup_{n=0}^{\infty} \{X_n = j\} | X_0 = i) > 0.$$

A state diagram for a Markov chain is a directed graph where the nodes represent the states and the edges represent possible one-step transitions. More precisely, the state diagram contains an edge from node i to node j if and only if $P_{ij} > 0$.

If $i, j \in \mathcal{S}$, then $i \to j$ if and only if the state diagram contains at least one directed path from i to j.



If such a path exists, we have:

$$P_{ij}^n \geq P_{i,k_1} \cdot P_{k_1,k_2} \cdots P_{k_{n-2},k_{n-1}} \cdot P_{k_{n-1},j} > 0.$$

Communicating states

States i and j communicate, denoted as $i \leftrightarrow j$, if $i \to j$ and $j \to i$.

The relation $\leftrightarrow$ is an equivalence relation. That is $\leftrightarrow$ satisfies the following properties:

- Reflexivity: $i \leftrightarrow i$.
- Symmetry: $i \leftrightarrow j$ if and only if $j \leftrightarrow i$.
- Transitivity: $i \leftrightarrow j$ and $j \leftrightarrow k$ implies $i \leftrightarrow k$.

Reflexivity follows since we always have $P_{ii}^0 = 1 > 0$. Symmetry follows directly from the definition.

Communicating states (cont.)

To prove transitivity we assume that $i \leftrightarrow j$ and $j \leftrightarrow k$.

Hence, in particular $i \rightarrow j$ and $j \rightarrow k$, implying that there exists $m, n \geq 0$ such that $P_{ij}^m > 0$ and $P_{jk}^n > 0$.

By the Chapman-Kolmogorov equations, we have:

$$P_{ik}^{m+n} = \sum_{r \in \mathcal{S}} P_{ir}^m P_{rk}^n \geq P_{ij}^m \cdot P_{jk}^n > 0.$$

Hence, by definition $i \rightarrow k$.

By a similar argument we can show that $k \rightarrow i$ as well.

Hence, we conclude that $i \leftrightarrow k$.

Communicating states (cont.)

Two states that communicate are said to be in the same (equivalence) class.

Two classes of states are either identical or disjoint.

PROOF: Assume that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ represent two equivalence classes, and assume that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. That is, there exists a state i such that $i \in \mathcal{A} \cap \mathcal{B}$.

Then choose $j \in \mathcal{A}$ and $k \in \mathcal{B}$ arbitrarily.

Now, $i, j \in \mathcal{A}$ implies that $i \leftrightarrow j$ and $i, k \in \mathcal{B}$ implies that $i \leftrightarrow k$.

Hence, by transitivity we also have $j \leftrightarrow k$. That is, j and k belong to the same equivalence class.

Since this holds for any $j \in \mathcal{A}$ and $k \in \mathcal{B}$, this implies that $\mathcal{A} = \mathcal{B}$ ■

The equivalence classes partition the state space $\mathcal{S}$ into a number of disjoint sets. A Markov chain is called irreducible if the number of equivalence classes is one.

Example 4.15

Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2\}$ and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

We then observe:

$$\text{Since } P_{01} = \tfrac{1}{2} > 0, \text{ it follows that } 0 \rightarrow 1$$
$$\text{Since } P_{10} = \tfrac{1}{2} > 0, \text{ it follows that } 1 \rightarrow 0$$
$$\text{Since } P_{12} = \tfrac{1}{4} > 0, \text{ it follows that } 1 \rightarrow 2$$
$$\text{Since } P_{21} = \tfrac{1}{3} > 0, \text{ it follows that } 2 \rightarrow 1$$

Hence, $0 \leftrightarrow 1$ and $1 \leftrightarrow 2$, and by transitivity $0 \leftrightarrow 2$ as well. Thus, the Markov chain is irreducible.

Example 4.15 (cont.)

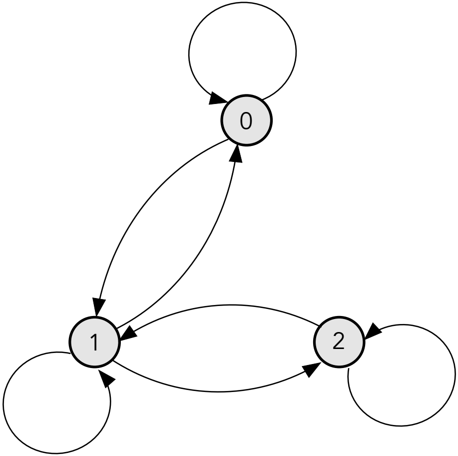


Figure: State diagram of an irreducible Markov chain with one class $\{0, 1, 2\}$

Example 4.16

A Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and matrix:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_{01} = P_{10} = \tfrac{1}{2}, \quad \Rightarrow \quad 0 \leftrightarrow 1$$

$$P_{0j} = P_{1j} = 0, \quad \Rightarrow \quad 0, 1 \not\to j, \quad j = 2, 3$$

$$P_{2i} = \tfrac{1}{4}, \quad \Rightarrow \quad 2 \to i, \quad i = 0, 1, 2, 3$$

$$P_{3i} = 0, \quad \Rightarrow \quad 3 \not\to i, \quad i = 0, 1, 2$$

The Markov chain has classes $\{0, 1\}$, $\{2\}$ and $\{3\}$, and is not irreducible.

Example 4.16 (cont.)

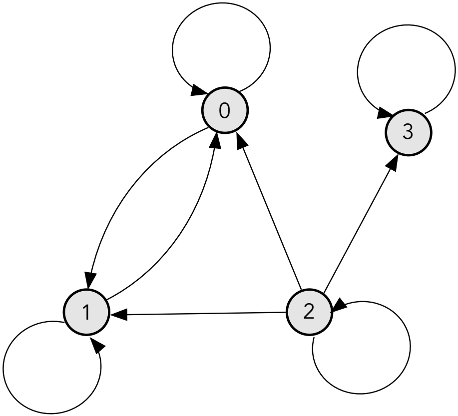


Figure: State diagram of a Markov chain with three classes $\{0, 1\}$, $\{2\}$ and $\{3\}$.

Recurrent and transient states

We consider the probabilities:

$$f_i = P\left(\bigcup_{r=1}^{\infty}\{X_r = i\}|X_0 = i\right), \quad i \in \mathcal{S}.$$

- State i is recurrent if $f_i = 1$.
- State i is transient if $f_i < 1$.

Assume that $X_0 = i$, and let N_i denote the number of times $X_n = i$.

- If i is recurrent, then $P(N_i = \infty|X_0 = i) = 1$.
- If i is transient, then $P(N_i = n|X_0 = i) = f_i^{n-1}(1 - f_i)$, $n = 1, 2, \ldots$.

If i is transient and $X_0 = i$, then $N_i|X_0 = i$ has a geometric distribution with $E[N_i|X_0 = i] = 1/(1 - f_i) < \infty$.

Proposition 4.1

Let $I_i^{(n)} = I(X_n = i)$, $n = 0, 1, \ldots$. We can then write:

$$N_i = \sum_{n=0}^{\infty} I_i^{(n)}$$

Hence, we have:

$$\begin{aligned} E[N_i | X_0 = i] &= \sum_{n=0}^{\infty} E[I_i^{(n)} | X_0 = i] \\ &= \sum_{n=0}^{\infty} P[X_n = i | X_0 = i] = \sum_{n=0}^{\infty} P_{ii}^{n} \end{aligned}$$

- State i is recurrent, if $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$.
- State i is transient, if $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$.

Corollary 4.2

If state i is recurrent, and $i \leftrightarrow j$, then state j is recurrent as well. Thus, recurrence is a class property.

PROOF: Since $i \leftrightarrow j$, there exists k and m such that $P_{ij}^k > 0$ and $P_{ji}^m > 0$.

Hence, for any $n = 1, 2, \ldots$ we have:

$$P_{jj}^{m+n+k} \geq P_{ji}^m \cdot P_{ii}^n \cdot P_{ij}^k.$$

Summing over all n, and using that i is recurrent, $P_{ij}^k > 0$ and $P_{ji}^m > 0$ we get:

$$\sum_{n=1}^{\infty} P_{jj}^{m+n+k} \geq P_{ji}^m \cdot P_{ij}^k \cdot \sum_{n=1}^{\infty} P_{ii}^n = \infty$$

Hence, we conclude that j is recurrent as well ■

Corollary 4.2 (cont.)

- If state i is transient and $i \leftrightarrow j$, then state j must also be transient. For if j were recurrent then, by Corollary 4.2, i would also be recurrent contradicting that i is transient. Thus, transience is a class property as well.
- If $\{X_n\}$ is a Markov chain with a finite state space, then at least one of the states must be recurrent. If $\{X_n\}$ is irreducible as well, then all states are recurrent.

Example 4.17

Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

It is easy to verify that $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$. Hence, the Markov chain is irreducible and thus all states must be recurrent ■

Example 4.17 (cont.)

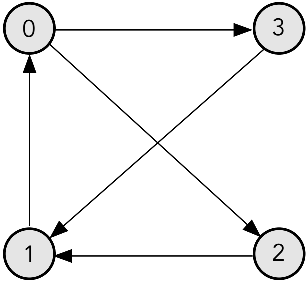


Figure: State diagram of an irreducible Markov chain with one class $\{0, 1, 2, 3\}$

Example 4.18

Consider a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and transition probability matrix:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

This chain has classes $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$.

The first two classes are recurrent and the third transient ■

Example 4.18 (cont.)

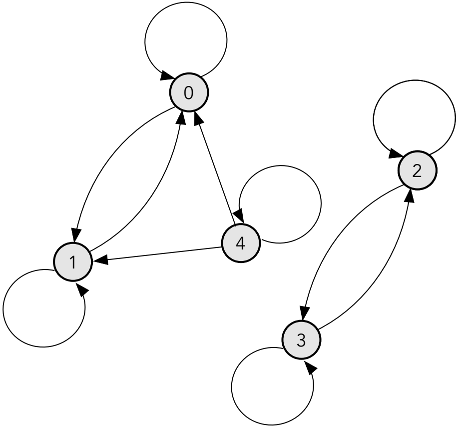


Figure: State diagram of a Markov chain with classes $\{0, 1\}$, $\{2, 3\}$ and $\{4\}$

Example 4.19 - Random walk

Consider a Markov chain with state space $\mathcal{S} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and where $0 < p < 1$ and:

$$P_{i,i+1} = p, \quad P_{i,i-1} = (1-p), \quad i \in \mathcal{S}.$$

It is obvious that $i \leftrightarrow j$ for all $i, j \in \mathcal{S}$. Hence, according to Corollary 4.2 all states are either recurrent or transient.

In order to check for recurrence, it is sufficient to check if $\sum_{n=1}^{\infty} P_{00}^n = \infty$.

Thus, we assume that $X_0 = 0$, and observe that in this case X_n is odd if n is odd, and X_n is even if n is even. Hence, since 0 is even, we have:

$$P_{00}^{2n-1} = 0, \quad n = 1, 2, \dots$$

$$P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{n!n!} [p(1-p)]^n, \quad n = 1, 2, \dots$$

Example 4.19 - Random walk (cont.)

We then use Stirling's formula for $n!$:

$$n! \approx n^{n+1/2} e^{-n} \sqrt{2\pi}$$

From this we get:

$$\frac{(2n)!}{n!n!} \approx \frac{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}}{(n^{n+1/2} e^{-n} \sqrt{2\pi})^2} = \frac{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}}{n^{2n+1} e^{-2n} (2\pi)} = \frac{2^{2n}}{\sqrt{n\pi}} = \frac{4^n}{\sqrt{n\pi}}$$

Hence:

$$P_{00}^{2n} = \frac{(2n)!}{n!n!} [p(1-p)]^n \approx \frac{(4p(1-p))^n}{\sqrt{n\pi}}$$

Example 4.19 - Random walk (cont.)

This implies that:

$$\sum_{n=1}^{\infty} P_{00}^{2n} \approx \sum_{n=1}^{\infty} \frac{(4p(1-p))^n}{\sqrt{n\pi}}$$

This series is divergent if and only if $p = \frac{1}{2}$.

Hence, the states are recurrent if and only if $p = \frac{1}{2}$.