STK2130 – Lecture 9

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Chapter 5.3.2 Definition of the Poisson Process

Definition (5.2)

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if the following four axioms hold:

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ *has independent increments.*

(iii) $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

Properties of the Poisson Process

Lemma (5.1)

Assume that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Then let $s > 0$, and define:

$$N_s(t) = N(s+t) - N(s), \qquad \text{for all } t \geq 0.$$

Then $\{N_s(t), t \geq 0\}$ is a Poisson process with rate λ as well.

Lemma (5.2)

If T_1 is the time of the first event of a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$, then:

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}, \quad t \geq 0.$$

That is $T_1 \sim exp(\lambda)$.

Properties of the Poisson Process (cont.)

Proposition (5.4)

Let $T_1, T_2, \ldots$ be the sequence of interarrival times of a Poisson process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$. That is, T_1 is the time of the first event, and

$$T_n = \text{The time between the } (n-1)\text{st and } n\text{th event}, \quad n = 2, 3, \ldots.$$

Then $T_1, T_2, \ldots$ are independent and identically distributed exponential random variables with rate λ.

PROOF: We have already shown that $T_1 \sim exp(\lambda)$. For T_2 we get:

$$\begin{aligned} P(T_2 > t | T_1 = s) &= P(N(s+t) - N(s) = 0 | T_1 = s) \\ &= P(N(s+t) - N(s) = 0) \qquad \text{by Axiom (ii)} \\ &= P(N_s(t) = 0) = e^{-\lambda t} \qquad \text{by Lemma 5.1 and Lemma 5.2.} \end{aligned}$$

The result follows by repeating this argument for $T_3, T_4, \ldots$ ∎

Properties of the Poisson Process (cont.)

We now introduce:

$$S_n = \sum_{i=1}^{n} T_i, \quad n = 1, 2, \ldots$$

By Proposition 5.4 S_n is a sum of independent identically distributed exponentially distributed variables. Hence it follows that:

$$S_n \sim gamma(n, \lambda), \quad n = 1, 2, \ldots$$

Properties of the Poisson Process (cont.)

Theorem (5.1)

Assume that $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$. Then $N(t)$ is a Poisson random variable with rate λt. That is:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 0, 1, 2, \ldots$$

PROOF: By Lemma 5.2 we have:

$$P(N(t) = 0) = e^{-\lambda t} = \frac{(\lambda t)^0}{0!} e^{-\lambda t}$$

In order to determine $P(N(t) = n)$ we condition on the event that $S_n = s$.

If $s > t$, we obviously have $P(N(t) = n | S_n = s) = 0$.

Properties of the Poisson Process (cont.)

We then consider the case where $0 < s \le t$, and get:

$$\begin{aligned} P(N(t) = n | S_n = s) &= P(T_{n+1} > t - s | S_n = s) \\ &= P(T_{n+1} > t - s) \qquad \text{since } T_{n+1} \text{ and } S_n \text{ are independent} \\ &= e^{-\lambda(t-s)}. \end{aligned}$$

Hence, it follows that:

$$\begin{aligned} P(N(t) = n) &= \int_0^t P(N(t) = n | S_n = s) f_{S_n}(s) ds \\ &= \int_0^t e^{-\lambda(t-s)} \cdot \frac{\lambda^n}{\Gamma(n)} s^{n-1} e^{-\lambda s} ds \;=\; \lambda^n e^{-\lambda t} \int_0^t \frac{s^{n-1}}{(n-1)!} ds \\ &= \lambda^n e^{-\lambda t} \cdot \Big|_0^t \frac{s^n}{n!} \;=\; \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 1, 2, \ldots \qquad \blacksquare \end{aligned}$$

Example 5.13

Suppose that people immigrate into a territory according to a Poisson process with rate $\lambda = 2$ per day.

(a) Find the probability there are 10 arrivals in the following week (of 7 days).

(b) Find the expected number of days until there have been 20 arrivals.

SOLUTION: (a) We recall that:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad n = 1, 2, \ldots$$

Hence, we get that:

$$P(N(7) = 10) = \frac{(2 \cdot 7)^{10}}{10!} e^{-2 \cdot 7}$$

$$= \frac{14^{10}}{10!} e^{-14} \approx 0.06628$$

Example 5.13 (cont.)

(b) We recall that:

$$S_n = \sum_{i=1}^{n} T_i = \text{The point of time of the } n\text{th arrival}$$

$$\sim gamma(n, \lambda).$$

Hence, we get:

$$E[S_n] = \sum_{i=1}^{n} E[T_i] = \frac{n}{\lambda}$$

In particular:

$$E[S_{20}] = \frac{20}{2} = 10.$$

Chapter 5.3.3 Further properties of Poisson Processes

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ. Each event can be classified as either a Type I event or a Type II event.

$$p = P(\text{A given event is of type I}),$$

$$1 - p = P(\text{A given event is of type II})$$

We then introduce:

$$N_1(t) = \text{ The number of type I events in } (0, t],$$

$$N_2(t) = \text{ The number of type II events in } (0, t].$$

NOTE: $N(t) = N_1(t) + N_2(t)$, and for $s < t$ we have:

$$(N_1(t) - N_1(s) | N(t) - N(s) = n) \sim Bin(n, p),$$

$$(N_2(t) - N_2(s) | N(t) - N(s) = n) \sim Bin(n, 1 - p)$$

Further properties of Poisson Processes (cont.)

From this we get that:

$$P(N_1(t) - N_1(s) = k) = \sum_{n=k}^{\infty} P(N_1(t) - N_1(s) = k | N(t) - N(s) = n) \cdot P(N(t) - N(s) = n)$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{[\lambda(t-s)]^n}{n!} e^{-\lambda(t-s)}$$

$$= \frac{(\lambda p(t-s))^k}{k!} e^{-\lambda p(t-s)} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)(t-s)]^{n-k}}{(n-k)!} e^{-\lambda(1-p)(t-s)}$$

$$= \frac{(\lambda p(t-s))^k}{k!} e^{-\lambda p(t-s)}$$

That is, $N_1(t) - N_1(s) \sim Po(\lambda p(t-s))$.

Similarly, $N_2(t) - N_2(s) \sim Po(\lambda(1-p)(t-s))$.

Further properties of Poisson Processes (cont.)

Proposition

$\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are both Poisson processes having rates λp and $\lambda(1-p)$ respectively, and the two processes are independent.

PROOF: Since $N(0) = 0$, it follows that $N_i(0) = 0$, $i = 1, 2$ as well.

Let $(s_1, t_1]$ and $(s_2, t_2]$ be disjoint. Since $\{N(t) : t \geq 0\}$ has independent increments, we have for $i = 1, 2$:

$$\begin{aligned}
&P(N_i(t_2) - N_i(s_2) = k | N_i(t_1) - N_i(s_1) = \ell) \\
&\quad = \sum_{n=k}^{\infty} P(N_i(t_2) - N_i(s_2) = k | N(t_2) - N(s_2) = n, N_i(t_1) - N_i(s_1) = \ell) \\
&\qquad \cdot P(N(t_2) - N(s_2) = n | N_i(t_1) - N_i(s_1) = \ell) \\
&\quad = \sum_{n=k}^{\infty} P(N_i(t_2) - N_i(s_2) = k | N(t_2) - N(s_2) = n) \cdot P(N(t_2) - N(s_2) = n) \\
&\quad = P(N_i(t_2) - N_i(s_2) = k)
\end{aligned}$$

Further properties of Poisson Processes (cont.)

Hence, $\{N_i(t) : t \geq 0\}$ have independent increments, $i = 1, 2$.

Moreover, we have:

$$\begin{aligned}
P(N_1(h) = 1) &= P(N_1(h) = 1 | N(h) = 1) P(N(h) = 1) \\
&+ P(N_1(h) = 1 | N(h) \geq 2) P(N(h) \geq 2) \\
&= p(\lambda h + o(h)) + o(h) \\
&= \lambda p h + o(h)
\end{aligned}$$

and:

$$P(N_1(h) \geq 2) \leq P(N(h) \geq 2) = o(h).$$

By similar arguments we get that:

$$\begin{aligned}
P(N_2(h) = 1) &= \lambda(1 - p)h + o(h) \\
P(N_2(h) \geq 2) &= o(h).
\end{aligned}$$

Further properties of Poisson Processes (cont.)

To show that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent we let $s < t$, and consider:

$$P[(N_1(t) - N_1(s) = k) \;\cap\; (N_2(t) - N_2(s) = \ell)]$$

$$= P[(N_1(t) - N_1(s) = k) \;\cap\; (N(t) - N(s) = k + \ell)]$$

$$= \binom{k+\ell}{k} p^k (1-p)^\ell \;\cdot\; \frac{[\lambda(t-s)]^{k+\ell}}{(k+\ell)!} e^{-\lambda(t-s)}$$

$$= \frac{(\lambda p(t-s))^k}{k!} e^{-\lambda p(t-s)} \;\cdot\; \frac{(\lambda(1-p)(t-s))^\ell}{\ell!} e^{-\lambda(1-p)(t-s)}$$

$$= P(N_1(t) - N_1(s) = k) \;\cdot\; P(N_2(t) - N_2(s) = \ell)$$

Hence, we conclude that $(N_1(t) - N_1(s))$ and $(N_2(t) - N_2(s))$ are independent for all $s < t$, implying that $\{N_1(t) : t \geq 0\}$ and $\{N_2(t) : t \geq 0\}$ are independent.

Example 5.14

If immigrants to area A arrive at a Poisson rate of $\lambda = 10$ per week, and if each immigrant is of English descent with probability $p = \frac{1}{12}$. What is the probability that no people of English descent will emigrate to area A during the month of February?

SOLUTION: By the previous proposition it follows that the number of Englishmen emigrating to area A during the month of February is Poisson distributed with mean:

$$\lambda \cdot \text{number of weeks in February} \cdot p = 10 \cdot 4 \cdot \frac{1}{12} = \frac{10}{3}.$$

Hence, we get:

$$P(\text{no people of English descent in February})$$

$$= \frac{(10/3)^0}{0!} e^{-10/3} = 0.0357$$

Example 5.15

We consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ where each event represents an offer. We introduce:

$$X_i = \text{ The size of the } i\text{th offer }, \quad , i = 1, 2, \ldots$$

We assume that $X_1, X_2, \ldots$ are non-negative, independent and identically distributed random variables with density $f(x)$. We assume that $f(x) > 0$ for all $x \geq 0$ and introduce:

$$\bar{F}(x) = P(X_i > x), \quad x \geq 0.$$

POLICY: Accept the first offer greater than some chosen number y, and define:

$$N_y(t) = \text{ The number of offers greater than } y \text{ in } (0, t], \quad t \geq 0.$$

Then $\{N_y(t) : t \geq 0\}$ is a Poisson process with rate $\lambda \bar{F}(y)$.

Example 5.15 (cont.)

If T_y is the time until the first offer which is greater than y, then:

$$T_y \sim exp(\lambda \bar{F}(y)),$$

and we get:

$$E[T_y] = \frac{1}{\lambda \bar{F}(y)}$$

The accepted offer is denoted by X. Thus, we know that $X > y$. Moreover, we must pay a waiting cost proportional to the waiting time T_y given by cT_y.

$$R(y) = \text{ Net return of the deal } = X - cT_y.$$

We then have:

$$E[R(y)] = E[X|X > y] - cE[T_y] = \int_y^\infty x \frac{f(x)}{\bar{F}(y)} dx - \frac{c}{\lambda \bar{F}(y)}.$$

Example 5.15 (cont.)

Thus, we may write:

$$E[R(y)] = \frac{\int_y^\infty xf(x)dx - c/\lambda}{\bar{F}(y)}$$

In order to find the value of y which maximizes $E[R(y)]$ we compute the derivative, and note that:

$$\frac{d}{dy}E[R(y)] = 0$$

if and only if:

$$\bar{F}(y) \cdot (-yf(y)) - (-f(y))\left[\int_y^\infty xf(x)dx - \frac{c}{\lambda}\right] = 0$$

or equivalently, since we have assumed that $f(y) > 0$ for all $y \geq 0$:

$$\left[\int_y^\infty xf(x)dx - \frac{c}{\lambda}\right] = \bar{F}(y)y = \int_y^\infty yf(x)dx$$

Example 5.15 (cont.)

The equation:

$$\left[\int_y^\infty xf(x)dx - \frac{c}{\lambda}\right] = \int_y^\infty yf(x)dx$$

can alternatively be written as:

$$\int_y^\infty (x-y)f(x)dx = \frac{c}{\lambda}$$

We denote the left-hand side of this equation by $\phi(y)$. Since we have assumed that $f(x) > 0$ for all $x \geq 0$, ϕ is a strictly decreasing function of y, and we have:

$$\phi(0) = \int_0^\infty (x-0)f(x)dx = E[X], \qquad \phi(\infty) = 0.$$

Thus, assuming that $E[X] > c/\lambda$, there exists a unique solution.

Example 5.15 (cont.)

Assume more specifically that:

$$f(x) = \frac{1}{\mu} e^{-x/\mu}, \quad x \geq 0.$$

Thus, $\bar{F}(x) = e^{-x/\mu}$, $E[X] = \mu$, and we have:

$$\phi(y) = \int_y^\infty (x - y) f(x) dx = \int_y^\infty (x - y) \frac{1}{\mu} e^{-x/\mu} dx$$

We then substitute $u = (x - y)$, and $du = dx$, and get:

$$\phi(y) = \int_0^\infty u \cdot \frac{1}{\mu} e^{-(u+y)/\mu} du$$

$$= e^{-y/\mu} \int_0^\infty u \cdot \frac{1}{\mu} e^{-u/\mu} du = \mu e^{-y/\mu}.$$

Example 5.15 (cont.)

Hence, the equation $\phi(y) = c/\lambda$ becomes:

$$\mu e^{-y/\mu} = \frac{c}{\lambda}$$

or equivalently:

$$-y/\mu = \ln(\frac{c}{\lambda\mu})$$

Thus, the optimal y-value, which we denote by y^* is given by:

$$y^* = -\mu \ln(\frac{c}{\lambda\mu}) = \mu \ln(\frac{\mu}{c/\lambda})$$

NOTE: Since we have assumed that $E[X] = \mu > c/\lambda$ it follows that $y^* > 0$.

Example 5.15 (cont.)

In the general case y^* satisfies:

$$\phi(y^*) = \int_{y^*}^{\infty} (x - y^*) f(x) dx = \frac{c}{\lambda}$$

We then consider:

$$\begin{aligned} E[R(y^*)] &= \frac{\int_{y^*}^{\infty} x f(x) dx - c/\lambda}{\bar{F}(y^*)} \\ &= \frac{1}{\bar{F}(y^*)} \left[\int_{y^*}^{\infty} (x - y^*) f(x) dx - \frac{c}{\lambda} + \int_{y^*}^{\infty} y^* f(x) dx \right] \\ &= \frac{1}{\bar{F}(y^*)} \left[\phi(y^*) - \frac{c}{\lambda} + y^* \bar{F}(y^*) \right] = y^* \end{aligned}$$

Thus, the optimal value y^* is also equal to the optimal expected net return.

Order statistics

Assume that $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed with density $f(y)$.

We then consider the corresponding order statistics obtained from $Y_1, Y_2, \ldots, Y_n$ by ordering these variables:

$$Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}$$

The joint density of the order statistics is given by:

$$f(y_{(1)}, y_{(2)}, \ldots, y_{(n)}) = n! \prod_{i=1}^{n} f(y_{(i)}), \qquad y_{(1)} < y_{(2)} < \cdots < y_{(n)}.$$

In particular, if $Y_i \sim \textit{uniform}(0, t)$, $i = 1, \ldots, n$, then:

$$f(y_{(1)}, y_{(2)}, \ldots, y_{(n)}) = \frac{n!}{t^n}, \qquad 0 < y_{(1)} < y_{(2)} < \cdots < y_{(n)} < t.$$

Order statistics (cont.)

NOTE: We observe that in the case where $Y_i \sim uniform(0, t)$, $i = 1, \ldots, n$, the joint density of $(Y_{(1)}, \ldots, Y_{(n)})$ is *constant* over the set:

$$\mathcal{Y} = \{(y_{(1)}, \ldots, y_{(n)}) : 0 < y_{(1)} < \cdots < y_{(n)} < t\}$$

This implies that:

$$\int_{\mathcal{Y}} f(y_{(1)}, \ldots, y_{(n)})\, dy_{(1)} \cdots dy_{(n)} = \frac{n!}{t^n} \cdot m(\mathcal{Y}) = 1,$$

where $m(\mathcal{Y})$ denotes the volume of the set $\mathcal{Y}$.

Hence, the volume of the set $\mathcal{Y}$ becomes:

$$m(\mathcal{Y}) = \frac{t^n}{n!}$$

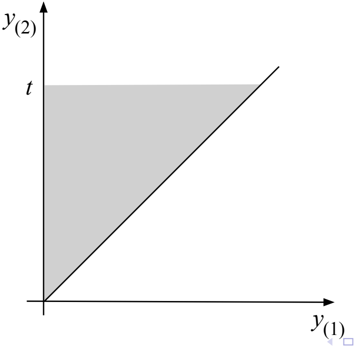
Order statistics (cont.)

In the special case where $n = 2$, the set:

$$\mathcal{Y} = \{(y_{(1)}, y_{(2)}) : 0 < y_{(1)} < y_{(2)} < t\}$$

is a *triangle* with area:

$$m(\mathcal{Y}) = \frac{t^2}{2!} = \frac{t^2}{2}$$



The scaled Dirichlet distribution

We recall that if $(Y_1, \ldots, Y_n) \sim Dirichlet(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, then the joint density of $(Y_1, \ldots, Y_n)$ is given by:

$$f(y_1, \ldots, y_n) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{n+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{n+1})} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1} (1 - \sum_{i=1}^{n} y_i)^{\alpha_{n+1} - 1}$$

for all $(y_1, \ldots, y_n)$ such that $y_i > 0$, for all i, and $\sum_{i=1}^{n} y_i < 1$.

We now let $t > 0$, and define $T_i = tY_i, \quad i = 1, \ldots, n$. Thus, the inverse transformation from $(t_1, \ldots, t_n)$ to $(y_1, \ldots, y_n)$ is given by:

$$y_i = \Psi_i(t_1, \ldots, t_n) = t_i / t, \quad i = 1, \ldots, n$$

From this it follows that the partial derivatives are given by:

$$\frac{\partial \Psi_i}{\partial t_i} = t^{-1}, \quad i = 1, \ldots, n.$$

while all other partial derivatives are zero.

The scaled Dirichlet distribution (cont.)

Hence, it follows that the Jacobi matrix is *diagonal*.

Thus, the Jacobi-determinant of the transformation is t^{-n}, and by the density transformation formula, the joint density of $T_1, \ldots, T_n$ is given by:

$$f(t_1, \ldots, t_n) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{n+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{n+1})} (\frac{t_1}{t})^{\alpha_1 - 1} \cdots (\frac{t_n}{t})^{\alpha_n - 1} (1 - \sum_{i=1}^{n} \frac{t_i}{t})^{\alpha_{n+1} - 1} \cdot t^{-n}$$

$$= \frac{\Gamma(\alpha_1 + \cdots + \alpha_{n+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{n+1})} t_1^{\alpha_1 - 1} \cdots t_n^{\alpha_n - 1} (t - \sum_{i=1}^{n} t_i)^{\alpha_{n+1} - 1} \cdot t^{1 - \sum_{i=1}^{n+1} \alpha_i}$$

for all $(t_1, \ldots, t_n)$ such that $t_i > 0$, for all i, and $\sum_{i=1}^{n} t_i < t$.

In particular, if $\alpha_1 = \cdots = \alpha_{n+1} = 1$, the joint density becomes:

$$f(t_1, \ldots, t_n) = \frac{n!}{t^n}$$

The scaled Dirichlet distribution (cont.)

NOTE: We observe that in the case where $\alpha_1 = \cdots = \alpha_{n+1} = 1$, the joint density of $(T_1, \ldots, T_n)$ is *constant* over the set:

$$\mathcal{T} = \{(t_1, \ldots, t_n) : t_1, \ldots, t_n > 0, \ \sum_{i=1}^{n} t_i < t\}$$

This implies that:

$$\int_{\mathcal{T}} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n = \frac{n!}{t^n} \cdot m(\mathcal{T}) = 1,$$

where $m(\mathcal{T})$ denotes the volume of the set $\mathcal{T}$.

Hence, the volume of the set $\mathcal{T}$ becomes:

$$m(\mathcal{T}) = \frac{t^n}{n!}$$

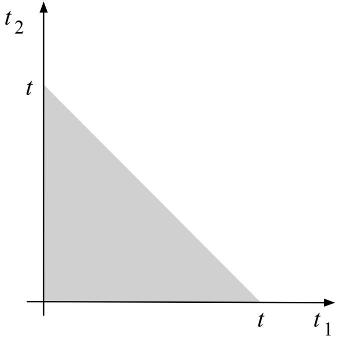
The scaled Dirichlet distribution (cont.)

In the special case where $n = 2$, the set:

$$\mathcal{T} = \{(t_1, t_2) : t_1 > 0,\ t_2 > 0,\ t_1 + t_2 < t\}$$

is a *triangle* with area:

$$m(\mathcal{T}) = \frac{t^2}{2!} = \frac{t^2}{2}$$



Chapter 5.3.4 Conditional distributions of the interarrival times and arrival times

Theorem (5.2 extended)

We consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ, and assume that $N(t) = n$. Then the interarrival times $T_1, \ldots, T_n$ has the following conditional joint density:

$$f(t_1, t_2, \ldots, t_n | N(t) = n) = \frac{n!}{t^n}, \qquad t_1, \ldots, t_n > 0, \; \sum_{i=1}^{n} t_i < t.$$

Furthermore, the arrival times $S_1, \ldots, S_n$ has the following conditional joint density:

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = \frac{n!}{t^n}, \qquad 0 < s_1 < s_2 < \cdots < s_n < t.$$

Conditional distributions (cont.)

PROOF: We start out by deriving the conditional joint distribution for $T_1, \ldots, T_n$ given that $N(t) = n$.

Let $t_1, \ldots, t_n > 0$ be such that $\sum_{i=1}^n t_i < t$, and let $h > 0$ be a sufficiently small number. Then we have:

$$P(t_1 < T_1 < t_1 + h, \ldots, t_n < T_n < t_n + h,\; N(t) = n)$$

$$\approx P(t_1 < T_1 < t_1 + h, \ldots, t_n < T_n < t_n + h,\; T_{n+1} > t - \sum_{i=1}^n t_i)$$

$$= \prod_{i=1}^n [e^{-\lambda t_i} - e^{-\lambda(t_i+h)}] \cdot e^{-\lambda(t - \sum_{i=1}^n t_i)}$$

$$= \prod_{i=1}^n e^{-\lambda t_i}[1 - e^{-\lambda h}] \cdot e^{-\lambda(t - \sum_{i=1}^n t_i)} = e^{-\lambda t}[1 - e^{-\lambda h}]^n$$

Conditional distributions (cont.)

Since $P(N(t) = n) = e^{-\lambda t}(\lambda t)^n/n!$, this implies that:

$$P(t_1 < T_1 \leq t_1 + h, \cdots, t_n < T_n \leq t_n + h | N(t) = n)$$
$$= \frac{e^{-\lambda t}[1 - e^{-\lambda h}]^n}{e^{-\lambda t}(\lambda t)^n/n!} = \frac{n!}{t^n} \cdot \left(\frac{1 - e^{-\lambda h}}{\lambda}\right)^n$$

From this it follows that:

$$\frac{1}{h^n} P(t_1 < T_1 \leq t_1 + h, \cdots, t_n < T_n \leq t_n + h | N(t) = n)$$
$$= \frac{n!}{t^n} \cdot \left(\frac{1 - e^{-\lambda h}}{\lambda h}\right)^n$$

Conditional distributions (cont.)

By using l'Hôpital's rule we get that:

$$\lim_{h \to 0} \frac{1 - e^{-\lambda h}}{\lambda h} = \lim_{h \to 0} \frac{\lambda e^{-\lambda h}}{\lambda} = 1$$

Hence, it follows that:

$$f(t_1, t_2, \ldots, t_n | N(t) = n) = \frac{n!}{t^n}, \qquad t_1, \ldots, t_n > 0, \ \sum_{i=1}^{n} t_i < t$$

Thus, we see that the conditional distribution of $T_1, \ldots, T_n$ given that $N(t) = n$ is a *scaled Dirichlet distribution* with parameters $\alpha_1 = \cdots \alpha_{n+1} = 1$, and scale factor t.

Conditional distributions (cont.)

In order to find the conditional distribution of $S_1, \ldots, S_n$ given that $N(t) = n$, we use that:

$$S_i = \sum_{j=1}^{i} T_j, \quad i = 1, \ldots, n.$$

Thus, the inverse transformation from $(s_1, \ldots, s_n)$ to $(t_1, \ldots, t_n)$ is given by:

$$t_i = \Psi_i(s_1, \ldots, s_n) = s_i - s_{i-1}, \quad i = 1, \ldots, n,$$

where we define $s_0 = 0$. From this it follows that the partial derivatives are given by:

$$\frac{\partial \Psi_i}{\partial s_i} = 1, \quad i = 1, \ldots, n,$$

$$\frac{\partial \Psi_i}{\partial s_{i-1}} = -1, \quad i = 2, \ldots, n,$$

while all other partial derivatives are zero.

Conditional distributions (cont.)

In particular since $\partial \Psi_i / \partial s_j = 0$ for all $j > i$, we conclude that the Jacobi matrix is *triangular*.

Thus, the Jacobi-determinant of the transformation is $1^n = 1$, and by the density transformation formula, the the conditional joint density of $S_1, \ldots, S_n$ given that $N(t) = n$ is given by:

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = \frac{n!}{t^n}, \qquad 0 < s_1 < s_2 < \cdots < s_n < t.$$

Thus, we see that the conditional joint distribution of $S_1, \ldots, S_n$ given that $N(t) = n$ is the same distribution as the joint distribution of the order statstics corresponding to n random variables uniformly distributed on the interval $(0, t)$ ■

Definition

A counting process $\{N(t) : t \geq 0\}$ *is said to be a non-homogeneous Poisson process with intensity function* $\lambda(t)$, $t \geq 0$, *if:*

(i) $N(0) = 0$

(ii) $\{N(t), t \geq 0\}$ *has independent increments.*

(iii) $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$

(iv) $P(N(t+h) - N(t) \geq 2) = o(h)$

We also introduce the mean value function $m(t)$ *defined by:*

$$m(t) = \int_0^t \lambda(u)du$$

The non-homogeneous Poisson Process (cont.)

Lemma (5.3)

If $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, then:

$$P(N(t) = 0) = e^{-m(t)}, \quad t \geq 0.$$

PROOF: Let $P_0(t) = P(N(t) = 0)$. Then:

$$\begin{aligned}
P_0(t+h) &= P(N(t+h) = 0) \\
&= P(N(t) = 0 \cap N(t+h) - N(t) = 0) \\
&= P(N(t) = 0) \cdot P(N(t+h) - N(t) = 0) \qquad \text{by Axiom (ii)} \\
&= P_0(t)(1 - \lambda(t)h + o(h)) \qquad \text{by Axiom (iii) and (iv)}
\end{aligned}$$

The non-homogeneous Poisson Process (cont.)

Hence, we get:

$$P_0(t+h) - P_0(t) = -\lambda(t)hP_0(t) + o(h)P_0(t)$$

Dividing by h and letting $h \to 0$ gives that:

$$P_0'(t) = \lim_{h\to 0} \frac{P_0(t+h) - P_0(t)}{h} = \lim_{h\to 0}[-\lambda(t)P_0(t) + \frac{o(h)P_0(t)}{h}] = -\lambda(t)P_0(t).$$

Alternatively, this can be expressed as:

$$\frac{P_0'(t)}{P_0(t)} = -\lambda(t)$$

We then integrate both sides of this equation:

$$\int_0^t \frac{P_0'(s)}{P_0(s)} ds = -\int_0^t \lambda(s)ds$$

The non-homogeneous Poisson Process (cont.)

On the left-hand side we substitute $u = P_0(s)$ and $du = P_0'(s)ds$, and get:

$$\int_{P_0(0)}^{P_0(t)} \frac{du}{u} = -\int_0^t \lambda(s)ds$$

The integration yields that:

$$\log(P_0(t)) - \log(P_0(0)) = -\int_0^t \lambda(s)ds$$

Since $P_0(0) = P(N(0) = 0) = 1$ it follows that:

$$P_0(t) = e^{-\int_0^t \lambda(s)ds} = e^{-m(t)}, \qquad \blacksquare$$

The non-homogeneous Poisson Process (cont.)

Corollary

If $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, and let T_1 be the time of the first event. Then we have:

$$P(T_1 > t) = P(N(t) = 0) = e^{-m(t)}, \quad t \geq 0.$$

Moreover, the density of T_1 is given by:

$$f_{T_1}(t) = \lambda(t)e^{-m(t)}, \quad t \geq 0.$$

PROOF: The first part follows immediate from Lemma 5.3, while the second part follows by differentiation ■

The non-homogeneous Poisson Process (cont.)

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process, and $s > 0$, we define:

$$N_s(t) = N(s + t) - N(s).$$

Lemma (5.4)

If $\{N(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda(t)$, then $\{N_s(t) : t \geq 0\}$ is a non-homogeneous Poisson process with intensity function $\lambda_s(t) = \lambda(s + t)$, $t \geq 0$.

PROOF: Similar to the proof of the corresponding result for homogeneous Poisson processes, i.e., Lemma 5.1. Simply replace λ by $\lambda(t)$ ■

The non-homogeneous Poisson Process (cont.)

The mean value function of $\{N_s(t) : t \geq 0\}$ is given by:

$$\begin{aligned} m_s(t) &= \int_0^t \lambda_s(u)du \\ &= \int_0^t \lambda(s+u)du \qquad \text{Subst.: } v = s+u,\ dv = du. \\ &= \int_s^{s+t} \lambda(v)dv \\ &= m(s+t) - m(s) \end{aligned}$$

Hence, we also have:

$$m_s(t-s) = m(t-s+s) - m(s) = m(t) - m(s).$$

The non-homogeneous Poisson Process (cont.)

Theorem

If $\{N(t), t \geq 0\}$ *is a non-homogeneous Poisson process with intensity function* $\lambda(t)$*, then:*

$$P(N(t) = n) = \frac{(m(t))^n}{n!} e^{-m(t)}, \quad t \geq 0, \quad n = 0, 1, 2, \ldots$$

PROOF: Induction with respect to n. By Lemma 5.3 the theorem holds for $n = 0$.

We then assume that we have shown that:

$$P(N(t) = n) = \frac{(m(t))^n}{n!} e^{-m(t)}, \quad t \geq 0,$$

and consider the probability $P(N(t) = n + 1)$.

The non-homogeneous Poisson Process (cont.)

In order to calculate this probability we condition on T_1, noting that if $s > t$, then obviously $P(N(t) = n + 1 | T_1 = s) = 0$.

$$\begin{aligned}
P(N(t) = n + 1) &= \int_0^t P(N(t) = n + 1 | T_1 = s)\, f_{T_1}(s) ds \\
&= \int_0^t P(N(t) = n + 1 | T_1 = s)\, \lambda(s) e^{-m(s)} ds \\
&= \int_0^t P(N(t) - N(s) = n | T_1 = s)\, \lambda(s) e^{-m(s)} ds \\
&= \int_0^t P(N(t) - N(s) = n)\, \lambda(s) e^{-m(s)} ds \qquad \text{(Indep. incr.)} \\
&= \int_0^t P(N_s(t - s) = n)\, \lambda(s) e^{-m(s)} ds
\end{aligned}$$

The non-homogeneous Poisson Process (cont.)

By Lemma 5.4 and the induction hypothesis it follows that:

$$P(N_s(t-s) = n) = \frac{(m_s(t-s))^n}{n!} e^{-m_s(t-s)}$$
$$= \frac{(m(t)-m(s))^n}{n!} e^{-(m(t)-m(s))}$$

By inserting this into the integral we get:

$$P(N(t) = n+1) = \int_0^t P(N_s(t-s) = n)\, \lambda(s) e^{-m(s)} ds$$
$$= \int_0^t \frac{(m(t)-m(s))^n}{n!} e^{-(m(t)-m(s))}\, \lambda(s) e^{-m(s)} ds$$

The non-homogeneous Poisson Process (cont.)

Simplifying the integrand yields:

$$P(N(t) = n+1) = \int_0^t \frac{(m(t) - m(s))^n}{n!} e^{-(m(t)-m(s))} \lambda(s) e^{-m(s)} ds$$

$$= \frac{e^{-m(t)}}{n!} \int_0^t (m(t) - m(s))^n \lambda(s) ds$$

We then substitute: $u = m(t) - m(s)$ and $du = -\lambda(s)ds$, and get:

$$P(N(t) = n+1) = \frac{e^{-m(t)}}{n!} \int_{m(t)}^0 (-u^n) du = \frac{e^{-m(t)}}{n!} \int_0^{m(t)} u^n du$$

$$= \frac{e^{-m(t)}}{n!} \cdot \Big|_0^{m(t)} \frac{u^{n+1}}{n+1} = \frac{(m(t))^{n+1}}{(n+1)!} e^{-m(t)}$$

which completes the induction proof ■