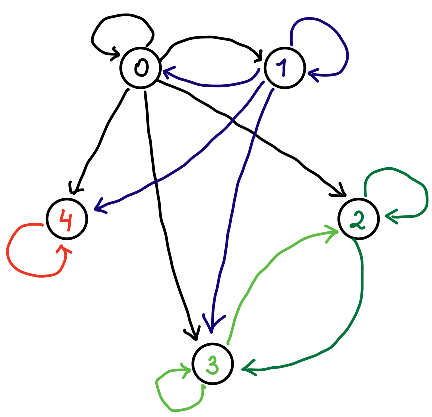
Solution to Mandatory Assignment

Problem 1 A Markov chain $X_0, X_1, X_2, \dots$ with state space $S = \{0,1,2,3,4\}$ is defined by the transition probability matrix

$$P := \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) The chain has three classes, $C_0 = \{0,1\}$, $C_1 = \{2,3\}$, $C_2 = \{4\}$. For each of these, discuss whether the class is recurrent or transient

The state diagram of the Markov chain is as follows:



- State 4 is absorbing $\Rightarrow$ the class $C_2 = \{4\}$ is recurrent.
- The chain stays in the class $C_1 = \{2,3\}$ whenever reaches it and cannot escape. The probability of returning to any state in C_1 given that the chain starts in C_1 is $1 \Rightarrow C_1$ is recurrent.
- If the chain leaves $C_0 = \{0,1\}$ (this happens with

positive probability), then it will never return to C_0 with probability 1 $\Rightarrow$ C_0 is transient.

Let T be the time until the chain enters C_1 or C_2 and define

$$\mu_i := \mathbb{E}[T \mid X_0 = i], \quad \text{for} \quad i \in C_0.$$

b) Explain why

$$\mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5}, \tag{1}$$

$$\mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5}. \tag{2}$$

Solve the equations (1) and (2) to obtain μ_0 and μ_1.

By the law of total expectation,

$$\mathbb{E}[T \mid X_0 = i] = \sum_{j=0}^{4} \mathbb{E}[T \mid X_0 = i, X_1 = j] \cdot \mathbb{P}(X_1 = j \mid X_0 = i). \tag{3}$$

Moreover, by the definition of T,

$$\mathbb{E}[T \mid X_0 = 0, X_1 = 0] = \mu_0 + 1,$$

$$\mathbb{E}[T \mid X_0 = 0, X_1 = 1] = \mu_1 + 1,$$

$$\mathbb{E}[T \mid X_0 = 1, X_1 = 0] = \mu_0 + 1,$$

$$\mathbb{E}[T \mid X_0 = 1, X_1 = 1] = \mu_1 + 1,$$

$$\mathbb{E}[T \mid X_0 = 0, X_1 = j] = 1, \quad j = 2, 3, 4,$$

$$\mathbb{E}[T \mid X_0 = 1, X_1 = j] = 1, \quad j = 2, 3, 4.$$

Plugging these expressions to (3), we immediately get (1) and (2).

Next, let us solve (1)-(2).

$$\begin{cases} \mu_0 = (\mu_0+1)\frac{1}{5} + (\mu_1+1)\frac{1}{5} + \frac{3}{5} \\ \mu_1 = (\mu_0+1)\frac{2}{5} + (\mu_1+1)\frac{1}{5} + \frac{2}{5} \end{cases} \Rightarrow \begin{cases} 5\mu_0 = \mu_0 + \mu_1 + 5 \\ 5\mu_1 = 2\mu_0 + \mu_1 + 5 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 4\mu_0 = \mu_1 + 5 \\ 2\mu_0 = 4\mu_1 - 5 \end{cases} \Rightarrow \begin{cases} \mu_1 + 5 = 8\mu_1 - 10 \\ \mu_0 = \frac{\mu_1+5}{4} \end{cases} \Rightarrow \begin{cases} \mu_1 = \frac{15}{7} \\ \mu_0 = \frac{25}{14} \end{cases}.$$

Let q_i be the probability that the chain ends up in state 4 given that $X_0 = i$, $i \in C_0$.

c) Find and explain equations for obtaining q_0 and q_1, and solve these equations.

Denote N – the first moment of reaching state 4. Then

$$q_0 = \mathbb{P}(N<\infty \mid X_0=0) = \sum_{i=0}^{4} \mathbb{P}(N<\infty \mid X_0=0, X_1=i)\,\mathbb{P}(X_1=i \mid X_0=0)$$

$$= \sum_{i=0}^{4} q_i\,\mathbb{P}(X_1=i \mid X_0=0) = [\, q_i=0,\ i=2,3;\ q_4=1 \,]$$

$$= q_0 \tfrac{1}{5} + q_1 \cdot \tfrac{1}{5} + \tfrac{1}{5}.$$

Similarly, $q_1 = q_0 \cdot \frac{2}{5} + q_1 \cdot \frac{1}{5} + \frac{1}{5}$. Thus

$$\begin{cases} q_0 = \frac{1}{5}q_0 + \frac{1}{5}q_1 + \frac{1}{5} \\ q_1 = \frac{2}{5}q_0 + \frac{1}{5}q_1 + \frac{1}{5} \end{cases} \Rightarrow \begin{cases} 4q_0 = q_1 + 1 \\ 4q_1 = 2q_0 + 1 \end{cases} \Rightarrow \begin{cases} q_1 = 4q_0 - 1 \\ 16q_0 - 4 = 2q_0 + 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} q_1 = 4q_0 - 1 \\ 14q_0 = 5 \end{cases} \Rightarrow \begin{cases} q_1 = \frac{3}{7} \\ q_0 = \frac{5}{14} \end{cases}$$

Let $s_{i,j}$ denote the expected number of visits to state j given that $X_0 = i$, $i, j \in C_0$.

d) Find and explain the equations for obtaining $s_{0,0}$, $s_{0,1}$, $s_{1,0}$, $s_{1,1}$ and solve these equations.

States $\{0,1\}$ form the class of all transient states. In Lecture 10 it was proved that the matrix

$$S = \begin{pmatrix} s_{0,0} & s_{0,1} \\ s_{1,0} & s_{1,1} \end{pmatrix}$$

can be found as

$$S = (I - P_T)^{-1},$$

where I is the identity matrix and P_T is the matrix that contains transition probabilities $P_{i,j}$, $i,j \in C_0$.

$$S = (I - P_T)^{-1} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \right)^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}^{-1}$$

$$\left(\begin{array}{cc|cc} \frac{4}{5} & -\frac{1}{5} & 1 & 0 \\ -\frac{2}{5} & \frac{4}{5} & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} \frac{4}{5} & -\frac{1}{5} & 1 & 0 \\ 0 & \frac{7}{10} & \frac{1}{2} & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} \frac{4}{5} & -\frac{1}{5} & 1 & 0 \\ 0 & 1 & \frac{5}{7} & \frac{10}{7} \end{array} \right) \sim$$

$$\sim \left(\begin{array}{cc|cc} \frac{4}{5} & 0 & \frac{8}{7} & \frac{2}{7} \\ 0 & 1 & \frac{5}{7} & \frac{10}{7} \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{10}{7} & \frac{5}{14} \\ 0 & 1 & \frac{5}{7} & \frac{10}{7} \end{array} \right),$$

i.e.

$$S = \begin{pmatrix} s_{0,0} & s_{0,1} \\ s_{1,0} & s_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{10}{7} & \frac{5}{14} \\ \frac{5}{7} & \frac{10}{7} \end{pmatrix}$$

Problem 2 Suppose that every second a router accepts letters from some alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$ with positive probabilities $p_1, \ldots, p_m$ respectively. The letters come independently.

A letter a_k at the moment n is coded by a number $x_k(n) \in \{1, \ldots, m\}$ (different letters are coded by different codes).

The coding algorithm is built in the following way. Assume that at the moment n the letter a_k arrives. Then $x_k(n+1)=1$. Letters encoded at the moment n by numbers $1, \dots, x_k(n)-1$ are re-encoded by numbers $2, \dots, x_k(n)$ respectively. The codes of other letters are left unchanged. We also assume that $x_k(0)=k$, $k=1,\dots,m$.

a) Define $X_n := (x_1(n), \dots, x_m(n))$. Prove that it is an ergodic Markov chain (i.e. irreducible, aperiodic and positive recurrent).

First, note that the state of $\{X_n\}$ at the time $n+1$ depends exclusively on its state at the moment n (and the letter arriving at time n which is independent of the past). Whence $\{X_n\}$ is Markovian. The state space S consists of all permutations of numbers $1, 2, \dots, m$, $|S| = m!$

Now, let us argue that $\{X_n\}$ is ergodic.

- $\{X_n\}$ is irreducible. Indeed, let us take two arbitrary states of $\{X_n\}$ (i.e. two arbitrary permutations of $1, 2, \dots, m$)

$$s=(s_1, s_2, \dots, s_m) \quad \text{and} \quad \tilde{s}=(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_m)$$

and assume that $X_n = s$. Assume that k_1 is such that $\tilde{s}_{k_1}=m$, k_2 is such that $\tilde{s}_{k_2}=m-1, \dots,$ k_m is such that $\tilde{s}_{k_m}=1$. Then, if

- letter a_{k_1} arrives at moment n,
- letter a_{k_2} arrives at moment $n+1$,

- letter a_{k_2} arrives at moment $n+2$,

. . .

- letter a_{k_m} arrives at moment $n+(m-1)$.

This happens with probability $\prod_{k=1}^{m} p_k > 0$ and in this case $X_{n+m} = \tilde{s}$. Thus

$$\mathbb{P}(X_{n+m} = \tilde{s} \mid X_n = s) \geq \prod_{k=1}^{m} p_k > 0$$

and the chain is irreducible.

- $\{X_n\}$ is aperiodic. Indeed, for any permutation $s \in \mathcal{S}$, $s = (s_1, \dots, s_m)$

$$\mathbb{P}(X_{n+1} = s \mid X_n = s) = p_{\ell_1} > 0,$$

where ℓ_1 is such that $s_{\ell_1} = 1$.

Whence $\{X_n\}$ cannot be periodic $\Rightarrow$ it is aperiodic.

- $\{X_n\}$ is positive recurrent. That follows from the fact that $\{X_n\}$ is irreducible and has finite state space.

b) Find the probability that a letter a_1 at the moment $n > 1$ is encoded by 1.

Letter a_1 is encoded by 1 at the moment $n > 1$ if and only if the letter a_1 arrived at the moment $n-1$. This happens with probability p_1. Whence

$$\mathbb{P}(x_1(n) = 1) = p_1.$$

c) Find the limit

$$\lim_{n\to\infty} \mathbb{P}(x_1(n)=1,\ x_2(n)=2).$$

First of all, $\{X_n\}$ is ergodic $\Longrightarrow$ this limit exists. Next,

$$\mathbb{P}(x_1(n+1)=1, x_2(n+1)=2) = \mathbb{P}(x_1(n)=1, x_2(n)=2, a_1 \text{ arrives at moment } n) +$$

$$+ \mathbb{P}(x_2(n)=1, a_1 \text{ arrives at moment } n) =$$

$$= \mathbb{P}(x_1(n)=1, x_2(n)=2)\,\mathbb{P}(a_1 \text{ arrives at moment } n) +$$

$$+ \mathbb{P}(x_2(n)=1)\,\mathbb{P}(a_1 \text{ arrives at moment } n)$$

Thus, if we denote $r_n := \mathbb{P}(x_1(n)=1, x_2(n)=2)$, we have that

$$r_{n+1} = r_n p_1 + p_1 p_2 .$$

Moving $n\to\infty$, we obtain:

$$r = r p_1 + p_1 p_2 \Longrightarrow r = \frac{p_1 p_2}{1-p_1}$$

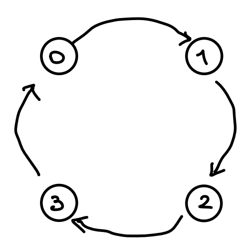
Problem 3 Give an example of a Markov chain $\{X_n\}$ and a subset $A \subset S$ for which the sequence

$$\{Y_n = \mathbb{1}_{\{X_n \in A\}}\}$$

a) is a Markov chain;

Let $\{X_n\}$ be a Markov chain with state space $\{0,1,2,3\}$ and the transition probability matrix,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



and initial distribution $\mathbb{P}(X_0 = i) = \frac{1}{4}$, $i = 0, 1, 2, 3$.

Let $\mathcal{A} = \{0, 2\}$. Then

- either $Y_{2n} = 0$, $Y_{2n+1} = 1$ for all $n \geq 0$ (with probability $\frac{1}{2}$),
- or $Y_{2n} = 1$, $Y_{2n+1} = 0$ for all $n \geq 0$ (with probability $\frac{1}{2}$),

i.e. $\{Y_n, n \geq 0\}$ is a Markov chain with transition probability matrix

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and initial distribution $\mathbb{P}(Y_0 = j) = \frac{1}{2}$, $j = 0, 1$.

b) is not a Markov chain.

Let $\{X_n\}$ be from the item a), but $\mathcal{A} = \{0, 1\}$. Then obviously

$$\mathbb{P}(Y_2 = 1 \mid Y_1 = 0, Y_0 = 0) = 1,$$

but

$$\mathbb{P}(Y_2 = 1 \mid Y_1 = 0) = \frac{1}{2},$$

i.e. the Markov property is violated.