12th March, 2022

STK2130

Mandatory assignment 1 of 1

Submission deadline

Thursday 31st March 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

For courses on bachelor level, you can choose between scanning handwritten notes or using a typesetting software for mathematics (e.g. LaTeX). Scanned pages must be clearly legible. For courses on master level the assignment must be written with a typesetting software for mathematics. It is expected that you give a clear presentation with all necessary explanations. The assignment must be submitted as a single PDF file. Remember to include any relevant programming code and resulting plots and figures, in the PDF-file.

All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, you may be asked to give an oral account.

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1. A Markov chain $X_0, X_1, X_2, \ldots$ with state space $\mathcal{S} = \{0, 1, 2, 3, 4\}$ is defined by the transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) The chain has three classes, $\mathcal{C}_0 = \{0, 1\}$, $\mathcal{C}_1 = \{2, 3\}$, $\mathcal{C}_2 = \{4\}$. For each of these classes discuss whether the class is *transient* or *recurrent*.

Let T be the time until the chain enters $\mathcal{C}_1$ or $\mathcal{C}_2$, and define:

$$\mu_i = E[T | X_0 = i], \quad \text{for } i \in \mathcal{C}_0.$$

b) Explain why:

$$\mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5} \tag{1}$$

$$\mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5} \tag{2}$$

Solve the equations (1) and (2) to obtain μ_0 and μ_1.

Let q_i be the probability that the chain ends up in state 4 given that $X_0 = i$, $i \in \mathcal{C}_0$.

c) Find and explain equations for obtaining q_0 and q_1, and solve these equations.

Let $s_{i,j}$ denote the expected number of visits to state j given that $X_0 = i$, $i, j \in \mathcal{C}_0$.

d) Find and explain equations for obtaining $s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}$, and solve these equations.

Problem 2. (*Coding by pile of books method*) Suppose that every second a router accepts letters from some alphabet $\mathcal{A} = \{a_1, ..., a_m\}$ with positive probabilities p_1,..., p_m respectively. The letters come independently.

A letter a_k at the moment n is coded by a number $x_k(n) \in \{1, ..., m\}$ (different letters are coded by different codes). The coding algorithm is built in the following way. Assume that at the moment n the letter a_k arrives. Then $x_k(n+1) = 1$. Letters encoded at the moment n by numbers 1,...,$(x_k(n) - 1)$, are re-encoded by numbers 2,...,$x_k(n)$, respectively. The codes of the other letters are left unchanged. We also assume that $x_k(0) = k$, $k = 1, ..., m$.

a) Define $X_n := (x_1(n), ..., x_m(n))$. Prove that it is an ergodic Markov chain (i.e. irreducible, aperiodic and positive recurrent). What is the state space of this Markov chain?

b) Find the probability that a letter a_1 at the moment $n > 1$ is encoded by 1.

c) Find the limit
$$\lim_{n\to\infty} \mathbb{P}\left(x_1(n) = 1,\ x_2(n) = 2\right).$$
Hint: let $r_n := \mathbb{P}\left(x_1(n) = 1,\ x_2(n) = 2\right)$. By analyzing the previous step, prove that $r_{n+1} = r_n p_1 + p_2 p_1$. Then move $n \to \infty$ and obtain the equation for $r := \lim_{n\to\infty} r_n$.

Problem 3. Give an example of a Markov chain $\{X_n\}$ and a subset $\mathcal{A} \subset \mathcal{S}$ for which the sequence $\{Y_n = \mathbb{1}_{\{X_n \in \mathcal{A}\}}\}$

a) is a Markov chain,

b) is not a Markov chain.